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Representations for the Three – Body T – Matrix, Scattering Matrices and Resolvent in Unphysical Energy Sheets

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Abstract. Explicit representations for the Faddeev components of the three – body T – matrix continued analytically into unphysical sheets of the energy Riemann surface are formulated. According to the representations, the T – matrix in unphysical sheets is explicitly expressed in terms of its components taken in the physical sheet only. The representations for the T – matrix are then used to construct similar representations for the analytic continuation of the three – body scattering matrices and the resolvent. Domains on unphysical sheets are described where the representations obtained can be applied. A method for finding three – body resonances based on the Faddeev differential equations is proposed.

1. Introduction

The topic of the paper is closely related to the problem of studying resonances in three – body quantum systems. The role played by such resonances is well known, for example, in the physics of nuclear reactions and in astrophysics. More generally, resonance is one of the most interesting phenomena in quantum scattering and the problem of definition and studying resonances attracts a lot of attention both from physicists and mathematicians. The literature on resonances is enormous and thus no attempt will be made here to present an exhaustive summary. For a history of the subject and a review see e. g. the books [1] – [9]. The main problems connected with a definition of resonance are explicitly emphasized by B. Simon in his survey [10]. In contrast to the usual (real) spectrum, the resonant one is not a unitary invariant of a (self – adjoint) operator and, thus, "no satisfactory definition of a resonance can depend only on the structure of a single operator on an abstract Hilbert space". Thus, a consideration of resonances is always connected with a rather concrete system, model, etc. supposing the presence of an extra structure like a "free" or "unperturbed"

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Hamiltonian or geometric features of the system or model concerned (see [10]).

The original idea of interpreting resonances in quantum mechanics as complex poles of the scattering matrix continued analytically into unphysical sheets of the energy plane goes back to G. Gamow [11]. In this interpretation, one actually compares the real dynamics of a system with some of its "free" dynamics (which is an extra structure in the sense of [10]). Here, resonances also manifest themselves as energy poles of the continued kernels of the wave operators (the scattering wave functions). For radially symmetric potentials, the interpretation of two – body resonances as poles of the analytic continuation of the scattering matrix has been entirely elaborated in terms of the Jost functions [12] (see e. g. [1], [5], [6], [7]).

Beginning with E. C. Titchmarsh [13] the resonances have also been interpreted as poles of the analytic continuations of the kernel of the Hamiltonian resolvent (or matrix elements of the resolvent between suitable states, see [1], [2]). In different forms this idea is realized in the papers [14] – [26] (see also the Refs. quoted in these papers and in the books [1] – [4]). In particular, such an interpretation became the basis for a perturbation theory for eigenvalues embedded in the continuous spectrum of N – body Hamiltonians and turning into resonances. This is a well studied subject now (see e. g. [3], [17], [19], [21], [26]). Another variant of a perturbation theory for the two – body resonances has been elaborated in [23] (see also [3]) for the case where the radius of the interaction tends to zero.

In the case where the interaction potentials are analytic functions of the coordinates, one can investigate resonances by the complex dilation method [16] (see also [2], [10]). The complex dilation makes it possible to rotate the continuous spectrum of the N – body Hamiltonian in such a way that certain sectors become accessible for observation in unphysical sheets neighboring the physical one [2]. Resonances situated in these sectors turn out to belong to the discrete spectrum of the transformed Hamiltonian. It should be noted that the resonances given by complex scaling are proved for a wide class of interactions to be not only poles of the resolvent but also poles of continued scattering amplitudes [20]. A number of rigorous results were obtained within the framework of this method (see e. g. the papers [16], [18], [20], [25], [27], [28] and the book [2]). Regarding the detailed of structure of the N – body scattering matrix and resolvent of the initial Hamiltonian continued into unphysical sheets, the complex dilation method gives not too large capacities.

A relation between analytic properties of the scattering matrix in the complex plane and the space – time behavior of wave packets has been studied (see the books [4] and [8] for references). Also the problem of completeness, normalization and orthogonalization of the resonance wave functions ("Gamow vectors"), i. e., solutions of the Schrödinger equation corresponding to the resonance energies had been widely discussed (see e. g. [5], [24], [29], [30]). Also attempts to interpret resonances and respective Gamow vectors in the framework of rigged Hilbert spaces [31] have been undertaken for a number of simple models (see [32], [33], [34]).

If the support of an interaction is compact, the resonances of the two – body system may be treated in the framework of the approach created by P. Lax and R. Phillips [35] (concerning its further development see [1] and [36] – [40]). A main advantage of this approach consists in the possibility of giving an elegant operator interpretation of the resonances. Precisely, it allows one describe resonances as the

discrete spectrum of a dissipative operator representing the generator of the compressed evolution (semi) group. Also, completeness of the resonance wave functions and an expansion theorem in the translationally invariant subspace [35] are naturally proved [37]. It should be noted however that the Lax – Phillips approach has strong restrictions on the domain of its applicability related in particular to the dimension of the configuration space of the system under consideration (the dimension has to be odd, and thus the N – body problem already with $N = 3$ cannot be treated). Up to now, the Lax – Phillips scheme has been realized in those scattering problems which generate the Riemann surfaces (though rather complicated) consisting only of two sheets of the complex energy plane (see [39], [40]). In the multichannel scattering problems with binary channels, this scheme has been partly realized in [41].

In the present paper we are concerned with the Faddeev approach [42] to the three – body problem. It is well known that many important conceptual and constructive results (see [42] – [46]) concerning the physical sheet in the three – body scattering problem have been obtained on the basis of the Faddeev equations [42] and their modifications. In particular, the structure of the resolvent and scattering operator was studied in detail, the completeness of the wave operators was proved and coordinate asymptotics of the scattering wave functions were investigated for rapidly decreasing as well as Coulomb interactions[1)] [42], [45], [46], [54]. Analogous results were obtained also for the singular interactions described by boundary conditions of various types [54] – [56]. On the basis of the Faddeev equations, various methods of investigation of concrete physical systems were developed in [46], [54], [57], [58]. Regarding the unphysical sheets, the situation with using these equations is rather different. Here, when studying a concrete three – body problem, one usually restricts oneself usually to developing a numerical algorithm to search for resonances in the unphysical sheets neighboring the physical one. A survey of physical approaches to a study of resonances in three – body nuclear systems based on the Faddeev equations can be found in [59] and [9].

The present work is devoted to extending the Faddeev approach [42], [46] to study the structure of the three – body T – matrix, resolvent and scattering matrices continued into unphysical sheets. We restrict ourselves to the case where the interaction potentials fall off in coordinate space not slower than exponentially. When constructing a theory of resonances in the two – body problem with such interactions one can use the coordinate as well as momentum representations. It is clear however that the analytic continuation of the Faddeev integral equations [42], [46] into unphysical sheets turns out to be a very difficult problem if the equations are written in the configuration space. The problem is that there exist noncompact (cylindrical) domains where the pair (two – body) potentials are translationally invariant and, therefore, do not decrease. At the same time the continued kernels of the equations increase exponentially and their solutions have to increase exponentially, too. Therefore, the integral

[1)] In the last decade, the new, more abstract approaches [47] – [51] (see also the literature cited in [51]) having no relation to the Faddeev – Yakubovsky techniques [42], [46] and [52], have been developed to prove the existence and asymptotic completeness of the N – body wave operators. In particular, in [51] such a proof is given for arbitrary N in the case where the pair interactions fall off at infinity like $r^{-\varrho}$, $\varrho > \sqrt{3}-1$, i. e., substantially slower than Coulomb potentials. Another approach to proving the absence of the singular continuous spectrum of the N – body Hamiltonians including the hard – core interactions has been worked out in [53].

terms diverge and the coordinate space equations do not make sense. On the other hand, the integral terms given in the momentum space can be considered as Cauchy type integrals admitting an explicit continuation. So, at least in the sense of distributions, a continuation of the momentum space Faddeev equations becomes a solvable problem. Actually, in the paper [60] (see also [59]), such a continuation into unphysical sheets neighboring the physical one has already been realized formally with the s – wave Faddeev equations corresponding to the rank 1 separable (finite-dimensional) pair potentials. In the present paper, we construct a continuation of the equations for the Faddeev components of the three – body T – matrix $T(z)$ in the case of sufficiently arbitrary pair potentials. We do this not only for the neighboring unphysical sheets but also for all those remote sheets of the three – body Riemann surface where it is possible to guide the spectral parameter (the energy z) around the two – body thresholds.

A central result of the paper consists in a substantiation of the existence of the analytic continuations (in the weak sense) of the Faddeev components $M_{\alpha\beta}(z)$, $\alpha, \beta = 1, 2, 3$, of the operator $T(z)$ and a construction of explicit (i. e., given in terms of the physical sheet only) representations for them in the unphysical sheets. These representations are found as a result of exactly solving the Faddeev equations for the matrix $M(z) = \{M_{\alpha\beta}(z)\}$ continued into unphysical sheets. Omitting details [see formula (7.34)], the representations read

$$M\big|_{\Pi_l} = M\big|_{\Pi_0} - \mathbf{Q}_l^{\dagger}\mathbf{J}^{\dagger}AS_l^{-1}\mathbf{J}\mathbf{Q}_l \tag{1.1}$$

where Π_l denotes the unphysical sheets enumerated by a (multi) index $l \neq 0$. The operator $\mathbf{Q}_l(z)$ and the "transposed" one $\mathbf{Q}_l^{\dagger}(z)$ are explicitly constructed from the matrix $M(z)$ taken in the physical sheet Π_0. The numerical matrix $A(z)$ is an entire function of $z \in \mathbb{C}$. By $S_l(z)$ [see (4.21)] we understand a truncation (depending essentially on l) of the total three – body scattering matrix $S(z)$. The operators $\mathbf{J}(z)$ and $\mathbf{J}^{\dagger}(z)$ realize a restriction of the kernels of the operators $\mathbf{Q}_l(z)$ and $\mathbf{Q}_l^{\dagger}(z)$ on energy shells respectively in the first and last momentum arguments so that the products $\mathbf{Q}_l^{\dagger}\mathbf{J}^{\dagger}$ and $\mathbf{J}\mathbf{Q}_l$ have half – on – shell kernels. Note that the structure of the representations (1.1) [(7.34)] for $M(z)\big|_{\Pi_l}$ is quite analogous to that of the representations found in the author's recent works [61] and [62] for the analytic continuation of the T – matrix in the multichannel scattering problems with binary channels. Representations for the analytic continuations of the three – body scattering matrices and resolvent follow immediately from the representations above for $M(z)\big|_{\Pi_l}$ [see Equations (8.1) and (9.1), respectively]. As follows from the representations (7.34), (8.1) and (9.1), the nontrivial (i. e., differing from the poles at points of the discrete spectrum of the three – body Hamiltonian) singularities of the T – matrix, scattering matrices and resolvent situated in the unphysical sheet Π_l are in fact singularities of the inverse truncated scattering matrix $S_l^{-1}(z)$. Therefore the resonances in the sheet Π_l, considered as poles of the T – matrix, scattering matrix and resolvent continued on Π_l, are actually those values of the energy z for which the matrix $S_l(z)$ has the eigenvalue zero. Of course, in analogy with a similar property of the two – body resonances this result can be considered as quite natural and rather expected.

Some basic results of the present work were announced in the report [63].

Let us describe shortly the structure of the paper. In Sec. 2, some general notations are given. Sec. 3 contains information on analytic properties of the two – body T – and scattering matrices which are used in subsequent sections. Sec. 4 is devoted to a description of properties of the matrix $M(z)$ and scattering matrices in the physical sheet of the energy z. In particular, the domains of the physical sheet where the half – on – shell kernels of $M(z)$ as well as the truncated scattering matrices $S_l(z)$ may be considered as holomorphic functions of z are described here. The kernels and matrices which are included in the explicit representations (7.34), (8.1) and (9.1) mentioned above are introduced. In Sec. 5 we specify the unphysical sheets included in a part $\Re$ of the three – body Riemann surface which we deal with in the paper. The analytic continuation of the Faddeev equations into unphysical sheets is made in Sec. 6. Sec. 7 is devoted to proving the validity of the explicit representations (7.34) for the analytic continuation of the matrix $M(z)$ into unphysical sheets of the surface $\Re$. In Sec. 8 we derive analogous representations [given by formulas (8.1)] for the scattering matrices, and in Sec. 9 the representations [given by formulas (9.1)] for the resolvent. In Sec. 10 we discuss the practical meaning of the results obtained. In particular we give here a sketch of a method to search for three – body resonances on the basis of Faddeev differential equations in the configuration space.

2. Notations

A three – body system in momentum space is considered. We enumerate the particles by the index $\alpha = 1, 2, 3$ and write k_α, p_α for the scaled relative momenta [46]. For instance

$$
\begin{aligned}
k_1 &= \left[\frac{\mathrm{m}_2+\mathrm{m}_3}{2\mathrm{m}_2\mathrm{m}_3}\right]^{1/2} \cdot \frac{\mathrm{m}_2\mathrm{p}_3-\mathrm{m}_3\mathrm{p}_2}{\mathrm{m}_2+\mathrm{m}_3}\,, \\
p_1 &= \left[\frac{\mathrm{m}_1+\mathrm{m}_2+\mathrm{m}_3}{2\mathrm{m}_1(\mathrm{m}_2+\mathrm{m}_3)}\right]^{1/2} \cdot \frac{(\mathrm{m}_2+\mathrm{m}_3)\mathrm{p}_1-\mathrm{m}_1(\mathrm{p}_2+\mathrm{p}_3)}{\mathrm{m}_1+\mathrm{m}_2+\mathrm{m}_3}
\end{aligned}
\tag{2.1}
$$

with m_α the masses and p_α the momenta of the particles. The movement of the center of mass of the system is assumed to be separated.

Expressions for the relative momenta k_α, p_α with $\alpha = 2, 3$ may be obtained from (2.1) by cyclic permutation of indices. Usually, we combine the relative momenta k_α, p_α into six – component vectors $P = \{k_\alpha, p_\alpha\}$. A choice of a certain pair $\{k_\alpha, p_\alpha\}$ fixes a Cartesian coordinate system in $\mathbb{R}^6$. Transition from one pair of the momenta to another one means a rotation in $\mathbb{R}^6$, $k_\alpha = c_{\alpha\beta}k_\beta + s_{\alpha\beta}p_\beta$, $p_\alpha = -s_{\alpha\beta}k_\beta + c_{\alpha\beta}p_\beta$, with coefficients $c_{\alpha\beta}$, $s_{\alpha\beta}$ depending on the particle masses only [46] such that $-1 < c_{\alpha\beta} < 0$, $s^2_{\alpha\beta} = = 1 - c^2_{\alpha\beta}$, $c_{\beta\alpha} = c_{\alpha\beta}$ and $s_{\beta\alpha} = -s_{\alpha\beta}$, $\beta \neq \alpha$.

In the momentum representation, the Hamiltonian H of the three – body system under consideration is defined by

$$
(Hf)(P) \;=\; P^2 f(P) + \sum_{\alpha=1}^{3} (v_\alpha f)(P)\,, \quad P^2 \;=\; k_\alpha^2 + p_\alpha^2\,, \quad f \in \mathcal{H}_0 \;\equiv\; L_2\big(\mathbb{R}^6\big)\,,
$$

with v_α the pair potentials which are assumed to be integral operators in k_α with the kernels $v_\alpha(k_\alpha, k'_\alpha)$. Hereafter, by the square of a vector of $\mathbb{R}^N$, $N = 3$ or $N = 6$, we understand the square of its modulus, e. g., $P^2 = |P|^2$.

For the sake of definiteness we suppose all the potentials v_α, $\alpha = 1, 2, 3$, to be local. This means that the kernel of v_α depends on the difference of the variables k_α and k'_α only: $v_\alpha(k_\alpha, k'_\alpha) = v_\alpha(k_\alpha - k'_\alpha)$. Actually, we consider two variants of the potentials v_α. In the first one, $v_\alpha(k)$ are holomorphic functions of the variable $k \in \mathbb{C}^3$ satisfying the estimate

$$|v_\alpha(k)| \;\le\; \frac{c}{(1+|k|)^{\theta_0}}\, e^{a_0|\mathrm{Im} k|} \quad \text{for all} \quad k \in \mathbb{C}^3 \tag{2.2}$$

for some $c > 0$, $a_0 > 0$ and $\theta_0 \in (3/2,\, 2)$. In the second variant, the potentials $v_\alpha(k)$ are holomorphic functions of k in the strip

$$W_{2b} \;=\; \{k :\; k \in \mathbb{C}^3,\; |\mathrm{Im}\, k| < 2b\}\,, \quad b \;>\; 0\,,$$

only and satisfy the condition

$$|v_\alpha(k)| \;\le\; \frac{c}{(1+|k|)^{\theta_0}} \quad \text{for all } \ k \ \text{ such that } \ |\,\mathrm{Im}\, k| \;<\; 2b\,. \tag{2.3}$$

In both variants the potentials v_α are supposed to be such that $v_\alpha(-k) = \overline{v_\alpha(k)}$ for any $k \in \mathbb{R}^3$. The latter condition guarantees self−adjointness of the Hamiltonian H on the set $\mathcal{D}(H) = \left\{f :\, \int_{\mathbb{R}^6} \left(1+P^2\right)^2 |f(P)|^2\, dP < \infty\right\}$ (see Theorem 1.1 of [42]).

Note that in the first variant, the requirement of holomorphy in all $\mathbb{C}^3$ and no more than exponential growth in $|\mathrm{Im}\, k|$ (2.2) mean that the potentials v_α have compact support in the coordinate space. In the second variant, the potentials $v_\alpha(k)$, rewritten in the coordinate representation, decrease exponentially.

By h_α, $(h_\alpha f)(k_\alpha) = k_\alpha^2 f(k_\alpha) + (v_\alpha f)(k_\alpha)$, we denote the Hamiltonian of the pair subsystem α. The operator h_α acts in $L_2(\mathbb{R}^3)$. Due to condition (2.2) or (2.3), its discrete spectrum $\sigma_d(h_\alpha)$ is negative and finite [1]. We enumerate the eigenvalues $\lambda_{\alpha,j} \in \sigma_d(h_\alpha)$, $\lambda_{\alpha,j} < 0$, $j = 1, 2, \ldots, n_\alpha$, $n_\alpha < \infty$, taking into account their multiplicities: each eigenvalue being repeated a number of times equal to its multiplicity. The maximum of these numbers is denoted by $\lambda_{\max}$, $\lambda_{\max} = \max_{\alpha,j} \lambda_{\alpha,j} < 0$. The notation $\psi_{\alpha,j}(k_\alpha)$ is used for the respective eigenfunctions.

By $\sigma_d(H)$ and $\sigma_c(H)$ we denote respectively the discrete and (absolutely) continuous components of the spectrum $\sigma(H)$ of the Hamiltonian H. Note that $\sigma_c(H) = [\lambda_{\min}, +\infty)$ with $\lambda_{\min} = \min_{\alpha,j} \lambda_{\alpha,j}$.

The notation H_0 is used for the operator of kinetic energy, $(H_0 f)(P) = P^2 f(P)$, while $R_0(z)$ and $R(z)$ stand for the resolvents of the operators H_0 and H, $R_0(z) = (H_0 - zI)^{-1}$ and $R(z) = (H - zI)^{-1}$. Here, I is the identity operator in $\mathcal{H}_0$.

Let $M_{\alpha\beta}(z) = \delta_{\alpha\beta} v_\alpha - v_\alpha R(z) v_\beta$, $\alpha, \beta = 1, 2, 3$, be the Faddeev components (these components were introduced in formulae (3.7) in paper [42] by L. D. FADDEEV) of the three−body T−matrix[2)] $T(z) = V - VR(z)V$ where $V = v_1 + v_1 + v_3$. The operators

[2)] Generally speaking, this operator does not possess a matrix structure. The traditional term "matrix" came from multi−channel scattering problems with binary channels where the operators defined analogously to $T(z)$ are indeed matrices.

$M_{\alpha\beta}(z)$ satisfy the Faddeev equations (the famous equations (3.9) of [42])

$$M_{\alpha\beta}(z) = \delta_{\alpha\beta}\mathbf{t}_\alpha(z) - \mathbf{t}_\alpha(z)R_0(z)\sum_{\gamma\neq\alpha} M_{\gamma\beta}(z), \quad \alpha = 1, 2, 3. \tag{2.4}$$

Here, the operator $\mathbf{t}_\alpha(z)$, $\alpha = 1, 2, 3$, has the kernel

$$\mathbf{t}_\alpha(P, P', z) = t_\alpha\big(k_\alpha, k'_\alpha, z - p_\alpha^2\big)\,\delta(p_\alpha - p'_\alpha), \tag{2.5}$$

where $t_\alpha(k, k', z)$ stands for the kernel of the operator $t_\alpha(z) = v_\alpha - v_\alpha r_\alpha(z)v_\alpha$ with $r_\alpha(z) = (h_\alpha - z)^{-1}$. The operator $t_\alpha(z)$, called the T – matrix (or transition operator) for the pair subsystem α, satisfies in turn the Lippmann – Schwinger equation

$$t_\alpha(z) = v_\alpha - v_\alpha r_0^{(\alpha)}(z)t_\alpha(z), \tag{2.6}$$

where $r_0^{(\alpha)}(z)$, $r_0^{(\alpha)}(z) = \big(h_0^{(\alpha)} - z\big)^{-1}$, is the resolvent of the kinetic energy operator $h_0^{(\alpha)}$ for the subsystem α, $\big(h_0^{(\alpha)}f_\alpha\big)(k_\alpha) = k_\alpha^2 f_\alpha(k_\alpha)$, $f_\alpha \in L_2\big(\mathbb{R}^3\big)$.

It is convenient to rewrite the system (2.4) in the matrix form

$$M(z) = \mathbf{t}(z) - \mathbf{t}(z)\mathbf{R}_0(z)\Upsilon M(z), \tag{2.7}$$

with $\mathbf{t}(z) = \mathrm{diag}\{\mathbf{t}_1(z), \mathbf{t}_2(z), \mathbf{t}_3(z)\}$ and $\mathbf{R}_0(z) = \mathrm{diag}\{R_0(z), R_0(z), R_0(z)\}$. By Υ we denote a 3×3 matrix with elements $\Upsilon_{\alpha\beta} = 1 - \delta_{\alpha\beta}$. $M(z)$ is a 3×3 operator matrix constructed from the components $M_{\alpha\beta}(z)$, $M = \{M_{\alpha\beta}\}$, $\alpha, \beta = 1, 2, 3$. The matrices M, $\mathbf{t}$, $\mathbf{R}_0$ and Υ are considered as operators in the Hilbert space $\mathcal{G}_0 = \bigoplus_{\alpha=1}^3 L_2\big(\mathbb{R}^6\big)$.

The matrix $M(z)$ obeys also an alternative variant of the Faddeev equations,

$$M(z) = \mathbf{t}(z) - M(z)\mathbf{R}_0(z)\Upsilon\mathbf{t}(z). \tag{2.8}$$

We shall also use the iterated equations (2.7) and (2.8),

$$M(z) = \sum_{k=0}^{m+n+1} \mathcal{Q}^{(k)} + \mathcal{Q}^{(m)}\mathbf{R}_0\Upsilon M\Upsilon\mathbf{R}_0\mathcal{Q}^{(n)}, \quad m, n \geq 0, \tag{2.9}$$

with

$$\mathcal{Q}^{(k)}(z) = \big(-\mathbf{t}(z)\mathbf{R}_0(z)\Upsilon\big)^k\mathbf{t}(z) = \mathbf{t}(z)\big(-\Upsilon\mathbf{R}_0(z)\mathbf{t}(z)\big)^k, \tag{2.10}$$

the iterations of the absolute term $\mathcal{Q}^{(0)}(z) = \mathbf{t}(z)$ in Eqs. (2.7) and (2.8).

The resolvent $R(z)$ is expressed in terms of the matrix $M(z)$ by the formula [46]

$$R(z) = R_0(z) - R_0(z)\Omega M(z)\Omega^\dagger R_0(z), \tag{2.11}$$

where $\Omega : \mathcal{G}_0 \to \mathcal{H}_0$ stands for the matrix – row $\Omega = (1,\ 1,\ 1)$ and $\Omega^\dagger = \Omega^* = (1,\ 1,\ 1)^\dagger$. Hereafter, the symbol "†" means transposition.

Throughout the paper we understand by $\sqrt{z-\lambda}$, $z \in \mathbb{C}$, $\lambda \in \mathbb{R}$, the main branch of the function $(z-\lambda)^{1/2}$. By $\hat{q}$ we usually denote the unit vector in the direction $q \in \mathbb{R}^N$, $\hat{q} = q/|q|$, and by S^{N-1} the unit sphere in $\mathbb{R}^N$, $\hat{q} \in S^{N-1}$. The inner

product in $\mathbb{R}^N$ is denoted by $(\cdot\,,\,\cdot)$. The notation $\langle\cdot\,,\,\cdot\rangle$ is used for inner products in infinite-dimensional Hilbert spaces.

Let $\mathcal{H}^{(\alpha,j)} = L_2(\mathbb{R}^3)$ and $\mathcal{H}^{(\alpha)} = \bigoplus_{j=1}^{n_\alpha} \mathcal{H}^{(\alpha,j)}$. By Ψ_α we denote the operator acting from $\mathcal{H}^{(\alpha)}$ into $\mathcal{H}_0$ defined by

$$(\Psi_\alpha f)(P) \;=\; \sum_{j=1}^{n_\alpha} \psi_{\alpha,j}(k_\alpha) f_j(p_\alpha)\,, \quad f \;=\; (f_1, f_2, \ldots, f_{n_\alpha})^\dagger\,.$$

The notation Ψ_α^* is used for the adjoint operator of Ψ_α. By Ψ we denote the block-diagonal matrix operator $\Psi = \operatorname{diag}\{\Psi_1, \Psi_2, \Psi_3\}$ which acts from $\mathcal{H}_1 = \bigoplus_{\alpha=1}^{3} \mathcal{H}^{(\alpha)}$ into $\mathcal{G}_0$ and by Ψ^* the adjoint operator of Ψ. Analogous to Ψ_α, Ψ_α^*, Ψ and Ψ^* we introduce operators Φ_α, Φ_α^*, Φ and Φ^*, which are obtained from the former by replacement of the eigenfunctions $\psi_{\alpha,j}(k_\alpha)$ with the "form factors" $\phi_{\alpha,j}(k_\alpha) = (v_\alpha \psi_{\alpha,j})(k_\alpha)$, $\alpha = 1,2,3$, $j = 1, 2, \ldots, n_\alpha$.

The two-body T-matrix $t_\alpha(z)$ is known (see §4 of [42] or §1 of Chapter III of [46]) to be an analytic operator-valued function of the variable $z \in \mathbb{C}\setminus[0,+\infty)$ having simple poles at the points $z \in \sigma_d(h_\alpha)$. Its kernel admits the representation

$$t_\alpha(k,k',z) \;=\; -\sum_{j=1}^{n_\alpha} \frac{\phi_{\alpha,j}(k)\overline{\phi_{\alpha,j}}(k')}{\lambda_{\alpha,j} - z} \;+\; \tilde{t}_\alpha(k,k',z)\,, \tag{2.12}$$

where $\tilde{t}_\alpha(k,k',z)$ is a holomorphic function in the variable $z \in \mathbb{C}\setminus[0,+\infty)$. Therefore

$$\mathbf{t}_\alpha(z) \;=\; -\,\Phi_\alpha \mathbf{g}_\alpha(z) \Phi_\alpha^* \;+\; \tilde{\mathbf{t}}_\alpha(z) \tag{2.13}$$

where $\tilde{\mathbf{t}}_\alpha(z)$ stands for the operator having the kernel $\tilde{t}_\alpha(k_\alpha, k'_\alpha, z - p_\alpha^2)\delta(p_\alpha - p'_\alpha)$. At the same time $\mathbf{g}_\alpha(z)$=diag$\{g_{\alpha,1}(z), \ldots, g_{\alpha,n_\alpha}(z)\}$ is a block-diagonal operator matrix whose elements $g_{\alpha,j}(z)$ are the operators in $\mathcal{H}^{(\alpha,j)}$ with the singular kernels $g_{\alpha,j}(p_\alpha, p'_\alpha, z) = \delta(p_\alpha - p'_\alpha)/(\lambda_{\alpha,j} - z + p_\alpha^2)$.

Below, we consider restrictions of different functions on the energy (or mass) shell

$$k \;=\; \sqrt{z}\,\hat{k}\,, \quad \hat{k} \in S^2\,, \tag{2.14}$$

in the two-body problem and on the energy (or mass) shells

$$P \;=\; \sqrt{z}\,\widehat{P}\,, \quad \widehat{P} \in S^5\,, \tag{2.15}$$

and

$$p_\alpha \;=\; \sqrt{z - \lambda_{\alpha,j}}\,\hat{p}_{\alpha,j}\,, \quad \hat{p}_{\alpha,j} \in S^2, \quad \alpha = 1,2,3, \;\; j = 1, 2, \ldots, n_\alpha\,, \tag{2.16}$$

in the problem of three particles. In the last case, the sets (2.15) and (2.16) are called respectively three-body and two-body energy shells.

Let $\mathcal{O}(\mathbb{C}^N)$ be a linear space of test functions represented by the Fourier transform of functions belonging to $C_0^\infty(\mathbb{R}^N)$ (we deal with $N = 3$ or $N = 6$ only). We mean $f \in \mathcal{O}(\mathbb{C}^N)$ if

$$f(q) \;=\; \int_{\mathbb{R}^N} dx \exp\{i(q_1 x_1 + \cdots + q_N x_N)\}\, f^\#(x) \quad \text{for some} \quad f^\# \in C_0^\infty(\mathbb{R}^N)\,.$$

where, $q = (q_1, q_2, \ldots, q_N) \in \mathbb{C}^N$, $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$. Every $f(q) \in \mathcal{O}(\mathbb{C}^N)$ is a holomorphic (entire) function of the variable $q \in \mathbb{C}^N$ satisfying the estimates

$$\left| \frac{\partial^{|m|}}{\partial q_1^{m_1} \cdot \ldots \cdot \partial q_N^{m_N}} f(q) \right| \;=\; c \cdot \exp(a|\operatorname{Im} q|)(1 + |q|)^{-\theta}\,,$$

where a is the radius of a ball centered at the origin and containing the support of the Fourier pre – image of f in $\mathbb{R}^N$, $|m| = m_1 + \cdots + m_N$, and $|\operatorname{Im} q| = \sqrt{\sum_{j=1}^N |\operatorname{Im} q_j|^2}$. For θ one can take an arbitrary positive number. The coefficient $c > 0$ depends on f, θ and $m = (m_1, \ldots, m_N)$.

A sequence of functions $\{f_j\}$ in the space $\mathcal{O}(\mathbb{C}^N)$ is said to be convergent to a function $f \in \mathcal{O}(\mathbb{C}^N)$ if for all $\theta > 0$ and $m_1, \ldots, m_N = 0, 1, 2, \ldots$ the equalities

$$\lim_{j\to\infty} \sup_{q\in\mathbb{C}^N} (1 + |q|)^\theta \exp(-a|\operatorname{Im} q|) \left| \frac{\partial^{|m|}}{\partial q_1^{m_1} \cdot \ldots \cdot \partial q_N^{m_N}} \big(f_j(q) - f(q)\big) \right| \;=\; 0$$

hold for some a not depending on m. Elements of the dual space $\mathcal{O}'(\mathbb{C}^N)$, the linear continuous functionals over $\mathcal{O}(\mathbb{C}^N)$, are usually called generalized functions or distributions[3)] (see e. g. [31], [64]).

Let $\mathrm{j}(z)$ be the operator which restricts functions $f(k)$, $k \in \mathbb{R}^3$, on the shell (2.14) at $z = E \pm i\,0$, $E > 0$, and continuing them if possible, on a domain of complex values of the energy z. On the set $\mathcal{O}(\mathbb{C}^3)$, the operator $\mathrm{j}(z)$ is defined by

$$\big(\mathrm{j}(z) f\big)(\hat{k}) \;=\; f\big(\sqrt{z}\,\hat{k}\big)\,. \tag{2.17}$$

Its kernel is a holomorphic generalized function (distribution) [64], $\mathrm{j}\big(\hat{k}, k', z\big) = \delta\big(\sqrt{z}\,\hat{k} - k'\big)$.

By $\mathrm{j}^\dagger(z)$ we denote the transposed operator of $\mathrm{j}(z)$. For any $\varphi \in L_2(S^2)$ this operator gives generalized function (distribution) over $\mathcal{O}(\mathbb{C}^3)$,

$$(\mathrm{j}^\dagger(z)\varphi)(k) \;=\; \int_{S^2} d\hat{k}'\, \delta\big(k - \sqrt{z}\,\hat{k}'\big)\, \varphi\big(\hat{k}'\big) \;=\; \frac{\delta\big(|k| - \sqrt{z}\big)}{z}\, \varphi\big(\hat{k}\big)\,, \tag{2.18}$$

i. e.,

$$(\mathrm{j}^\dagger(z)\varphi, f) \;=\; \int_{S^2} d\hat{k} f\big(\sqrt{z}\,\hat{k}\big) \varphi\big(\hat{k}\big)\,, \quad f \in \mathcal{O}(\mathbb{C}^3)\,. \tag{2.19}$$

Let $\mathrm{J}_{\alpha,j}(z)$, $\alpha = 1, 2, \ldots,$ $j = 1, 2, \ldots,$ be the operator of restriction on the shell (2.16). Its action on $\mathcal{O}(\mathbb{C}^3)$ is defined by

$$(\mathrm{J}_{\alpha,j}(z) f)(\hat{p}_\alpha) \;=\; f\big(\sqrt{z - \lambda_{\alpha,j}}\;\hat{p}_\alpha\big)\,, \quad \alpha \;=\; 1,\, 2,\, 3, \quad j \;=\; 1,\, 2, \ldots, n_\alpha\,.$$

3) Note that, in fact, we could consider narrower classes of distributions over spaces of test functions holomorphic only in those domains [described in terms of the energy shells (2.15) and (2.16)] where the scattering matrices and some half – on – shell kernels encountered below may be continued into the physical energy sheet. However the results [representations (7.34), (8.1) and (9.1)] do not depend on such a choice.

The operators $\mathrm{J}_{\alpha,j}(z)$ have the kernels

$$\mathrm{J}_{\alpha,j}(\hat{p}_\alpha, p'_\alpha, z) \;=\; \delta\big(\sqrt{z-\lambda_{\alpha,j}}\,\hat{p}_\alpha - p'_\alpha\big)\,.$$

By $\mathrm{J}_0(z)$ we denote the operator of restriction on the shell (2.15). On $\mathcal{O}\big(\mathbb{C}^6\big)$ this operator is defined by $\big(\mathrm{J}_0(z)f\big)\big(\widehat{P}\big) = f\big(\sqrt{z}\widehat{P}\big)$. The notations $\mathrm{J}^\dagger_{\alpha,j}(z)$ and $\mathrm{J}^\dagger_0(z)$ are used for the respective transposed operators. Their actions are defined similarly to (2.18), (2.19) by

$$\big(\mathrm{J}^\dagger_{\alpha,j}(z)\varphi\big)(p_\alpha) \;=\; \int_{S^2} d\hat{p}'_\alpha \delta\big(p_\alpha - \sqrt{z-\lambda_{\alpha,j}}\,\hat{p}'_\alpha\big)\varphi(\hat{p}'_\alpha)\,, \quad \varphi \in \widehat{\mathcal{H}}^{(\alpha,j)},$$

$$\big(\mathrm{J}^\dagger_0(z)\varphi\big)(P) \;=\; \int_{S^5} d\widehat{P}'\delta\big(P - \sqrt{z}\widehat{P}'\big)\varphi\big(\widehat{P}'\big)\,, \quad \varphi \in \widehat{\mathcal{H}}_0\,,$$

where $\widehat{\mathcal{H}}^{(\alpha,j)} \equiv L_2\big(S^2\big)$ and $\widehat{\mathcal{H}}_0 \equiv L_2\big(S^5\big)$. The generalized functions $\mathrm{J}^\dagger_{\alpha,j}(z)\varphi$ and $\mathrm{J}^\dagger_0(z)\varphi$ are elements of the spaces $\mathcal{O}'\big(\mathbb{C}^3\big)$ and $\mathcal{O}'\big(\mathbb{C}^6\big)$ of distributions over $\mathcal{O}\big(\mathbb{C}^3\big)$ and $\mathcal{O}\big(\mathbb{C}^6\big)$, respectively.

The operators $\mathrm{J}_{\alpha,j}$ and $\mathrm{J}^\dagger_{\alpha,j}$ are then combined into the block–diagonal matrices $\mathrm{J}^{(\alpha)}(z) = \mathrm{diag}\{\mathrm{J}_{\alpha,1}(z), \ldots, \mathrm{J}_{\alpha,n_\alpha}(z)\}$ and $\mathrm{J}^{(\alpha)\dagger}(z) = \mathrm{diag}\,\big\{\mathrm{J}^\dagger_{\alpha,1}(z), \ldots, \mathrm{J}^\dagger_{\alpha,n_\alpha}(z)\big\}$. The latter are used to construct the operators

$$\mathrm{J}_1(z) \;=\; \mathrm{diag}\,\big\{\mathrm{J}^{(1)}(z), \mathrm{J}^{(2)}(z), \mathrm{J}^{(3)}(z)\big\}\,, \quad \mathrm{J}^\dagger_1(z) \;=\; \mathrm{diag}\,\big\{\mathrm{J}^{(1)\dagger}(z), \mathrm{J}^{(2)\dagger}(z), \mathrm{J}^{(3)\dagger}(z)\big\}\,.$$

The action of $\mathrm{J}^{(\alpha)}(z)$ and $\mathrm{J}_1(z)$ on elements of the spaces $\mathcal{O}^{(\alpha)} = \times_{\alpha=1}^{n_\alpha}\mathcal{O}^{(\alpha,j)}$, $\mathcal{O}^{(\alpha,j)} \equiv \equiv \mathcal{O}\big(\mathbb{C}^3\big)$ and $\mathcal{O}_1 = \times_{\alpha=1}^{3}\mathcal{O}^{(\alpha)}$, respectively, can be understood from the definition of the operators $\mathrm{J}_{\alpha,j}(z)$. The operators $\mathrm{J}^{(\alpha)\dagger}(z)$ act from $\widehat{\mathcal{H}}^{(\alpha)} \equiv \oplus_{j=1}^{n_\alpha}\widehat{\mathcal{H}}^{(\alpha,j)}$ to the space $\mathcal{O}^{(\alpha)\prime}$ of distributions over $\mathcal{O}^{(\alpha)}$ and the operator $\mathrm{J}^\dagger_1(z)$ from $\widehat{\mathcal{H}}_1 = \oplus_{\alpha=1}^{3}\widehat{\mathcal{H}}^{(\alpha,j)}$ to the space $\mathcal{O}'_1$ of distributions over $\mathcal{O}_1$.

Finally, we use the block–diagonal operator 3×3–matrices

$$\mathbf{J}_0(z) \;=\; \mathrm{diag}\{\mathrm{J}_0(z), \mathrm{J}_0(z), \mathrm{J}_0(z)\}\,, \quad \mathbf{J}^\dagger_0(z) \;=\; \mathrm{diag}\,\big\{\mathrm{J}^\dagger_0(z), \mathrm{J}^\dagger_0(z), \mathrm{J}^\dagger_0(z)\big\}$$

constructed from the operators $\mathrm{J}_0(z)$ and $\mathrm{J}^\dagger_0(z)$, respectively, as well as the operators $\mathbf{J}(z) = \mathrm{diag}\{\mathrm{J}_0(z), \mathrm{J}_1(z)\}$ and $\mathbf{J}^\dagger(z) = \mathrm{diag}\,\big\{\mathrm{J}^\dagger_0(z), \mathrm{J}^\dagger_1(z)\big\}$. The actions of these operators is clear from the definitions of the operators J_0, J_1, $\mathrm{J}^\dagger_0$ and $\mathrm{J}^\dagger_1$. In particular, the operator $\mathbf{J}^\dagger(z)$ acts from the space $\widehat{\mathcal{G}}_0 = \oplus_{\alpha=1}^{3}\widehat{\mathcal{H}}_0$ to the space $\times_{\alpha=1}^{3}\mathcal{O}'\big(\mathbb{C}^6\big)$.

The identity operators in the spaces $\widehat{\mathcal{H}}_0$, $\widehat{\mathcal{G}}_0$, $\widehat{\mathcal{H}}_1$ and $\widehat{\mathcal{H}}_0 \oplus \widehat{\mathcal{H}}_1$ are denoted by $\hat{I}_0$, $\hat{\mathbf{I}}_0$, $\hat{I}_1$ and $\hat{\mathbf{I}}$, respectively.

3. Analytic continuation of the two–body T– and scattering matrices

In this section we recall some analytic properties of pair T–matrices the knowledge of which will be necessary for posing the three–body problem below. Note that these

properties are well known (see e. g. [1], [7] and [59]) for a wide class of the potentials v_α. As a matter of fact, we give here only an explicit representation for the two – body T – matrix in an unphysical sheet which is a particular case of the explicit representations constructed in the author's work [61] (see Theorem 2 in [61] and comments to it) for a more general situation of analytic continuation of the T – matrix on unphysical sheets in the multichannel problem with binary channels. On the other hand, using this simple example related to the two – body problem we can demonstrate the main features of the scheme which we apply later in the three – body problem.

Throughout the section we shall consider a fixed two – body subsystem of the three – body system concerned. Therefore, its index will be omitted in the notations. Statements will be given for the first variant of potentials (2.2). If necessary, assertions corresponding to the second variant (2.3) will be written in parentheses.

According to Eq. (2.6) the kernel $t(k,k',z)$ of the pair T – matrix satisfies the integral equation

$$t(k,k',z) \;=\; v(k,k') \;-\; \int_{\mathbb{R}^3} dq\, \frac{v(k,q)t(q,k',z)}{q^2-z}\,. \tag{3.1}$$

All the dependence of $t(k,k',z)$ on z is determined by the integral term of the equation. The latter, with respect to the variable z, represents a Cauchy type integral. Integrals of this kind appear in the Faddeev equations (2.4) as well and they are all of the form

$$\Phi(z) \;=\; \int_{\mathbb{R}^N} dq\, \frac{f(q)}{\lambda+q^2-z} \tag{3.2}$$

where $N=3$ or $N=6$ and $\lambda \le 0$.

Let us describe some properties of the function $\Phi(z)$, supposing that f is a holomorphic function of the variable $q \in \mathbb{C}^N$. We also assume that f satisfies the estimate $|f(q)| \le c\,M(q)$ where $M(q)>0$ and $\int_{S^{N-1}} d\hat{q}\, M(q) \le c\,\frac{\exp(a|\operatorname{Im} q|)}{(1+|q|)^\theta}$ for some $c>0$ and $\theta \in (N-2,N-1)$.

Let $\Re_\Phi$ be the Riemann surface of the function

$$\zeta(z) \;=\; \begin{cases} (z-\lambda)^{1/2}\,, & N \text{ odd}\,,\\ \ln(z-\lambda)\,, & N \text{ even}\,. \end{cases}$$

The surface $\Re_\Phi$ arises as a result of pasting the sheets Π_l representing copies of the complex plane $\mathbb{C}$ cut along the ray $[\lambda,+\infty)$. The sheet Π_l is identified with a branch of the function $\zeta(z)$. If $\zeta(z)=(z-\lambda)^{1/2}$, we suppose for the main branch that $l=0$, otherwise $l=1$. If N is even, then as l we take a number of the function $\ln(z-\lambda)$ branch, $\ln(z-\lambda)=\ln|z-\lambda|+i\varphi+i2\pi l$ where $\varphi=\arg(z-\lambda)$, so that $z-\lambda=|z-\lambda|\exp(i\varphi)$ and $\varphi\in[0,2\pi)$.

Lemma 3.1. *The function $\Phi(z)$ is holomorphic in the complex plane $\mathbb{C}$ cut along the ray $[\lambda,+\infty)$ and admits the analytic continuation on the Riemann surface $\Re_\Phi$ given by*

$$\Phi(z)\big|_{\Pi_l} \;=\; \Phi(z) \;-\; \pi i l\big(\sqrt{z-\lambda}\,\big)^{N-2} \int\limits_{S^{N-1}} d\hat{q}\, f\big(\sqrt{z-\lambda}\,\hat{q}\big) \tag{3.3}$$

and the estimates

$$\|\Phi(z)\| \;\le\; c\,\|f\|_\theta (1+|z|)^{-\nu'},$$

$$\int\limits_{S^{N-1}} d\hat{q} f\big(\sqrt{z-\lambda}\hat{q}\big) \;\le\; c\,\|f\|_\theta (1+|z|)^{-\theta/2} \exp\big(a\big|\operatorname{Im}\sqrt{z-\lambda}\,\big|\big)$$

with $\|f\|_\theta = \sup_{q\in\mathbb{C}^N} M^{-1}(q)|f(q)|$ *hold for any* $\nu' < (\theta-(N-2))/2$.

For a proof see [61].

Using the relations (3.3) one can easily obtain representations for the analytic continuations of the kernels $r_0(k,k',z)$ and $R_0(P,P',z)$. The Riemann surface $\Re^{(2)}$ of the kernel $r_0(z)$ coincides with $\Re_\Phi$ corresponding to $\zeta(z) = z^{1/2}$, since, in this case, N is odd ($N=3$). For $R_0(z)$, the Riemann surface is logarithmic, $\zeta(z) = \ln(z)$ (N even, $N=6$). The continuation of the kernels $r_0(k,k',z)$ and $R_0(P,P',z)$ in z is understood in the sense of generalized functions (distributions) over $\mathcal{O}(\mathbb{C}^3)$ and $\mathcal{O}(\mathbb{C}^6)$, respectively. In the example with $r_0(z)$, we consider the continuation of the bilinear form $\Phi(z) = (r_0(z)f_1, f_2) \equiv \int_{\mathbb{R}^3} dq\, \frac{f_1(q)f_2(q)}{q^2-z}$, $f_1, f_2 \in \mathcal{O}(\mathbb{C}^3)$. By Lemma 3.1 one gets immediately

$$r_0(z)\big|_{\Pi_1} \;=\; r_0(z) \,+\, \mathrm{a}_0(z)\mathrm{j}^\dagger(z)\mathrm{j}(z)\,, \qquad \mathrm{a}_0(z) \;=\; -\,\pi i\sqrt{z}\,, \tag{3.4}$$

$$R_0(z)\big|_{\Pi_l} \;=\; R_0(z) \,+\, A_0(z)\,l\,\mathrm{J}_0^\dagger(z)\mathrm{J}_0(z)\,, \quad A_0(z) \;=\; -\,\pi i z^2, \quad l \;=\; \pm1,\, \pm2,\, \ldots\,. \tag{3.5}$$

Using Lemma 3.1 one can immerse the equation (3.1) in a suitable Banach space and show then that for the first variant of potentials (2.2), the integral operator $vr_0(z)$ can be continued analytic in z as a compact operator on the whole Riemann surface $\Re^{(2)}$. Regarding the second variant of the potentials (2.3), the existence of such a continuation into the unphysical sheet is guaranteed only for the domain $\Pi_1 \cap \mathcal{P}_b$,

$$\mathcal{P}_b \;=\; \left\{ z:\;\; \operatorname{Re} z > -b^2 + \frac{1}{4b^2}\,(\operatorname{Im} z)^2 \right\} \tag{3.6}$$

bounded by the parabola $\operatorname{Im}\sqrt{z} = b$ inside of which the function $v\big(\sqrt{z}\,(\hat{k}-\hat{k}')\big)$ is holomorphic in z for arbitrary $\hat{k},\, \hat{k}' \in S^2$. According to Eq. (3.4) the product $vr_0(z)\big|_{\Pi_1}$ may be written as $vr_0(z)\big|_{\Pi_1} = vr_0(z) + \mathrm{a}_0(z)v\mathrm{j}^\dagger(z)\mathrm{j}(z)$. Thus, the continued equation (3.1) for $t'(z) = t(z)\big|_{\Pi_1}$ has the form

$$t'(z) \;=\; v - vr_0(z)t'(z) \;-\; \mathrm{a}_0(z)v\,\mathrm{j}^\dagger(z)\,\mathrm{j}(z)\,t'(z)\,. \tag{3.7}$$

Let us transfer the term $vr_0(z)t'(z)$ to the left−hand side of Eq. (3.7) and invert (at $z \notin \sigma_d(h)$) the operator $[I + vr_0(z)]^{-1}$ using the relation $[I+vr_0(z)]^{-1}v = t(z)$. This leads to the expression

$$t'(z) \;=\; t(z) \;-\; \mathrm{a}_0(z)t(z)\mathrm{j}^\dagger(z)\mathrm{j}(z)t'(z) \tag{3.8}$$

for $t'(z)$ in terms of the half – on – shell kernel $t'\big(\sqrt{z}\,\hat{k}, k', z\big)$, the first argument of which belongs to the energy – shell (2.14). Now to find $\mathrm{j}t'$ we apply j to the both parts of Eq. (3.8) and then transfer a term including $\mathrm{j}(z)t(z)$ to the left – hand side. Then we get

$$s(z)\mathrm{j}(z)t'(z) \;=\; \mathrm{j}(z)t(z) \tag{3.9}$$

where $s(z)$, $z \in \Pi_0$, is the two – body scattering matrix (cf., e. g., formula (7.70) of [46]),

$$s(z) \;=\; \hat{I} \,+\, \mathrm{a}_0(z)\mathrm{j}(z)t(z)\mathrm{j}^\dagger(z)\,, \tag{3.10}$$

$s(z) : L_2\big(S^2\big) \to L_2\big(S^2\big)$, with $\mathrm{a}_0(z) = -\pi i\sqrt{z}$ and $\hat{I}$ is the identity operator in $L_2\big(S^2\big)$.

The absolute term $(\mathrm{j}t)\big(\hat{k}, \hat{k}', z\big)$ of Eq. (3.9), considered as a function of the first argument $\hat{k} \in S^2$, is an element of $L_2\big(S^2\big)$ at $z \notin \sigma_d(h)$. The operator $\mathrm{j}t\mathrm{j}^\dagger$ on the right in (3.10) is a compact operator in $L_2\big(S^2\big)$ for $z \notin \sigma_d(h)$. Therefore, on the domain of analyticity $\Pi_0 \setminus \sigma_d(h)$ $\big(\mathcal{P}_b \cap \Pi_0 \setminus \sigma_d(h)\big)$ of the operator – valued function $(\mathrm{j}t\mathrm{j}^\dagger)(z)$, one can apply to Eq. (3.9) the Fredholm analytic alternative [65] (see [61]). This means that, with exception of a countable set σ_{res} having no limit points in $\mathbb{C} \setminus \overline{\sigma(h)}$ $\big(\mathcal{P}_b \setminus \overline{\sigma(h)}\big)$, the inverse operator $[s(z)]^{-1}$ exists and $\mathrm{j}t'(z) = [s(z)]^{-1}\mathrm{j}t(z)$. As a result we come to the following statement which is in fact a one – channel variant of Theorem 2 of [61].

Theorem 3.2. *The two – body T – matrix $t(z)$ admits analytic continuation in the variable z on the sheet Π_1 (on the domain $\mathcal{P}_b \cap \Pi_1$) as a bounded operator in $L_2\big(\mathbb{R}^3\big)$. The result of the continuation $t(z)\big|_{\Pi_1}\Big(t(z)\big|_{\mathcal{P}_b\cap\Pi_1}\Big)$ is expressed by T – and S – matrices taken in the physical sheet as*

$$t(z)\big|_{\Pi_1} \;=\; t(z) \,-\, \mathrm{a}_0(z)\tau(z) \tag{3.11}$$

where $\tau(z) \;=\; \big(t\mathrm{j}^\dagger s^{-1}\mathrm{j}t\big)(z)$. The kernel $t(k, k', z)\big|_{\Pi_1}$ is a holomorphic function of the variables $k, k' \in \mathbb{C}^3$ and $z \in \Pi_1 \setminus \big(\sigma_{\mathrm{res}} \cup \sigma_d(h)\big)$ $\big(k$ and k' belonging to W_b and $z \in \mathcal{P}_b \cap \Pi_1 \setminus \big(\sigma_{\mathrm{res}} \cup \sigma_d(h)\big)\big)$.

On the physical sheet Π_0, the pair T – matrix $t(z)$ admits the representation (2.12) including the formfactors ϕ_j, $j = 1, 2, \ldots, n$. It follows from the Lippmann – Schwinger equation for ϕ_j that

$$\phi_j(k) \;=\; -\int_{\mathbb{R}^3} dq\, v(k, q)\,\frac{1}{q^2 - \lambda_j}\,\phi_j(q)\,, \quad \lambda_j \;<\; 0\,, \tag{3.12}$$

that the formfactor $\phi_j(k)$ admits an analytic continuation in k on $\mathbb{C}^3$ (on W_{2b}) and that, at the same time, it satisfies the type (2.2) estimate where one has to replace θ_0 by a number θ, $1 < \theta < \theta_0$, which can be taken arbitrarily close to θ_0 [42]. Hence the eigenfunction

$$\psi_j(k) \;=\; -\,\frac{\phi_j(k)}{k^2 - \lambda_j} \tag{3.13}$$

of h admits also such an analytic continuation on $\mathbb{C}^3$ (on W_{2b}) with the exception of the set $\{k \in \mathbb{C}^3 : k^2 = \lambda_j\}$ where $\psi_j(k)$ has singularities $\big($turning for $k = \sqrt{z}\,\hat{k}$, $\hat{k} \in S^2$, into a pole in z at $z = \lambda_j\big)$.

The regular summand $\tilde{t}(k, k', z)$ of the kernel of $t(z)$ is a holomorphic function in the variables $k, k' \in \mathbb{C}^3$, $z \in \Pi_0$ ($k, k' \in W_b$, $z \in \mathcal{P}_b \cap \Pi_0$). It admits the estimate

$$|\tilde{t}(k, k', z)| \;<\; c\,(1 + |k - k'|)^{-\theta} \cdot \exp[a(|\mathrm{Im}\,k| + |\mathrm{Im}\,k'|)]$$

with arbitrary $\theta \in (1, \theta_0)$.

Regarding the operator $t(z)\big|_{\Pi_1}$, it follows from Eq. (3.11) that the points $z \in \sigma_d(h)$ become, generally speaking, poles of the first order of this operator. One can easily check however that if the eigenvalue $\lambda \in \sigma_d(h)$ is simple, then the respective singularities of both summands of (3.11) compensate each other and the there is no pole of $t(z)\big|_{\Pi_1}$ at $z = \lambda$. It follows from the Fredholm analytic alternative [65] for Eq. (3.9) that the poles of $t(z)\big|_{\Pi_1}$ at $z \in \sigma_{\mathrm{res}}$ are of a finite order. It is also easy to show that if $\mathcal{A}(\hat{k})$ is a nontrivial solution of the equation

$$s(z)\mathcal{A} \;=\; 0 \tag{3.14}$$

at some $z \in \sigma_{\mathrm{res}}$, $z \notin \sigma_d(h)$, then the Schrödinger equation has at this z a nontrivial (resonance) solution $\psi^{\#}_{\mathrm{res}}$. The asymptotics of such a solution written in configuration space is exponentially increasing,

$$\psi^{\#}_{\mathrm{res}}(x) \underset{x\to\infty}{=} (\mathcal{A}(-\hat{x}) + o(1))\,\frac{\mathrm{e}^{-i\sqrt{z}\,|x|}}{|x|}\,, \quad x \in \mathbb{R}^3\,.$$

The function $\psi^{\#}_{\mathrm{res}}(x)$ is a so-called Gamow vector corresponding to the resonance at the energy z (see e.g. [6], [33], [8]). The function $\mathcal{A}\big(\hat{k}\big)$ gives sense to the breakup amplitude of the resonance state.

The formula for the analytic continuation of the scattering matrix into the unphysical sheet Π_1 (on the set $\mathcal{P}_b \cap \Pi_1$) follows immediately from Eq. (3.11) (see [61]),

$$s(z)\big|_{\Pi_1} \;=\; \mathcal{E}\,[s(z)]^{-1}\mathcal{E}\,, \tag{3.15}$$

where $\mathcal{E}$ stands for the inversion in $L_2\big(S^2\big)$, $(\mathcal{E}f)\big(\hat{k}\big) = f\big(-\hat{k}\big)$.

Utilizing the representation (3.11) one can easily get an explicit representation in terms of the physical sheet as well for the analytic continuation on Π_1 (on $\mathcal{P}_b \cap \Pi_1$) of the resolvent $r(z)$:

$$r(z)\big|_{\Pi_l} \;=\; r \,+\, \mathrm{a}_0(I - rv)\mathrm{j}^{\dagger}s^{-1}\mathrm{j}(I - vr)\,. \tag{3.16}$$

The continuation is understood again in the sense of generalized functions (distributions) over $\mathcal{O}\big(\mathbb{C}^3\big)$. This means that one has to continue the bilinear form

$$\Phi(z) \;=\; \big(r(z)f_1, f_2\big) \;\equiv\; \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dk'\, f_2(k) r(k, k', z) f_1(k')\,, \quad f_1,\, f_2 \in \mathcal{O}\big(\mathbb{C}^3\big)\,.$$

4. The matrix $M(z)$ and the three – body scattering matrices in the physical sheet

4.1. Faddeev equations

At the beginning, we recall briefly some principal properties [42], [46] of the Faddeev equations (2.7) for the matrix $M(z)$ and of the kernels $M_{\alpha\beta}(P,P',z)$ themselves for real arguments $P,P' \in \mathbb{R}^6$. To formulate these properties we reproduce here the following definition from [42], §5.

An operator – valued function $\mathcal{Q}_{\alpha\beta}(z) : \mathcal{H}_0 \to \mathcal{H}_0$, $\alpha,\beta = 1,2,3$, of the variable $z \in \mathbb{C}$, is said to be a function of type $\mathcal{D}_{\alpha\beta}$ if it admits the representation

$$
\begin{aligned}
\mathcal{Q}_{\alpha\beta}(z) \;&=\; \mathcal{F}_{\alpha\beta}(z) + \Phi_\alpha \mathbf{g}_\alpha(z)\mathcal{I}_{\alpha\beta}(z) + \mathcal{J}_{\alpha\beta}(z)\mathbf{g}_\beta(z)\Phi_\beta^* \\
&\quad + \Phi_\alpha \mathbf{g}_\alpha(z)\mathcal{K}_{\alpha\beta}(z)\mathbf{g}_\beta(z)\Phi_\beta^* .
\end{aligned}
\tag{4.1}
$$

The operator – valued functions $\mathcal{F}_{\alpha\beta}(z)$ and $\mathcal{I}_{\alpha\beta}(z)$ from $\mathcal{H}_0$ into $\mathcal{H}_0$ and $\mathcal{H}^{(\alpha)}$, respectively, and $\mathcal{J}_{\alpha\beta}(z) : \mathcal{H}^{(\beta)} \to \mathcal{H}_0$ and $\mathcal{K}_{\alpha\beta}(z) : \mathcal{H}^{(\beta)} \to \mathcal{H}^{(\alpha)}$ are called components of the function $\mathcal{Q}_{\alpha\beta}(z)$. If $\mathcal{Q}_{\alpha\beta}(z)$ is an integral operator, then its kernel is of the type $\mathcal{D}_{\alpha\beta}$.

Let

$$
\mathcal{N}(P,\theta) \;=\; \sum_{\substack{\alpha,\beta,\\ \alpha\neq\beta}} (1+|p_\alpha|)^{-\theta}(1+|p_\beta|)^{-\theta} .
$$

A function $\mathcal{Q}(z)$ of type $\mathcal{D}_{\alpha\beta}$ is said to be a function of class $\mathcal{D}_{\alpha\beta}(\theta,\mu)$ if its components $\mathcal{F}_{\alpha\beta}$, $\mathcal{I}_{\alpha\beta}$, $\mathcal{J}_{\alpha\beta}$ and $\mathcal{K}_{\alpha\beta}$ are integral operators and for the kernels $\mathcal{F}_{\alpha\beta}(P,P',z)$ at P, P', ΔP, $\Delta P' \in \mathbb{R}^6$, the estimates

$$
|\mathcal{F}_{\alpha\beta}(P,P',z)| \;\le\; c\mathcal{N}(P,\theta)\big(1+{p'_\beta}^2\big)^{-1} , \tag{4.2}
$$

$$
\begin{aligned}
&|\mathcal{F}_{\alpha\beta}(P+\Delta P, P'+\Delta P', z+\Delta z) - \mathcal{F}(P,P',z)| \\
&\le\; c\mathcal{N}(P,\theta)\big(1+{p'_\beta}^2\big)^{-1}(|\Delta P|^\mu + |\Delta P'|^\mu + |\Delta z|^\mu)
\end{aligned}
\tag{4.3}
$$

are valid for a certain $c > 0$ and, at the same time the kernels $\mathcal{I}_{\alpha,j;\beta}\big(p_\alpha, P', z\big)$, $\mathcal{J}_{\alpha;\beta,k}\big(P, p'_\beta, z\big)$ and $\mathcal{K}_{\alpha,j;\beta,k}\big(p_\alpha, p'_\beta, z\big)$ satisfy the inequalities obtained from (4.2) and (4.4) by taking, respectively, $k_\alpha = 0$, $k'_\beta = 0$ or simultaneously $k_\alpha = 0$ and $k'_\beta = 0$.

Let $\mathcal{Q}^{(n)}(z)$ be the iteration (2.10) of the absolute term of Eq. (2.7). In contrast to $\mathcal{Q}^{(0)}(z) = \mathbf{t}(z)$, the kernels of the operators $\mathcal{Q}^{(n)}(z)$, beginning already with $n = 1$, do not include δ – functions. Moreover, it follows from the representation (2.13) for $\mathbf{t}_\alpha(z)$ explicitly manifesting a contribution of the discrete spectrum of the pair subsystems, that the matrix elements $\mathcal{Q}^{(n)}_{\alpha\beta}(z)$, $\alpha,\beta = 1,2,3$, of the operators $\mathcal{Q}^{(n)}(z)$ for $n \ge 1$ are actually functions of type $\mathcal{D}_{\alpha\beta}$. Their components $\mathcal{F}^{(n)}_{\alpha\beta}(z)$, $\mathcal{I}^{(n)}_{\alpha\beta}(z)$, $\mathcal{J}^{(n)}_{\alpha\beta}(z)$ and $\mathcal{K}^{(n)}_{\alpha\beta}(z)$ at $z \in \mathbb{C} \setminus [\lambda_{\min}, +\infty)$ are bounded operators depending on z analytically. In the case of the potentials (2.2) and (2.3) at $z \notin [\lambda_{\min}, +\infty)$, the Hölder index μ of smoothness of their kernels with respect to the variables P, P', p_α and p'_β is equal to 1. If $n \le 3$, then, as $\operatorname{Im} z \to 0$ with $\operatorname{Re} z \in [\lambda_{\min}, +\infty)$, the kernels $\mathcal{F}^{(n)}_{\alpha\beta}$, $\mathcal{I}^{(n)}_{\alpha,j;\beta}$,

$\mathcal{J}^{(n)}_{\alpha;\beta,k}$, and $\mathcal{K}^{(n)}_{\alpha,j;\beta,k}$ have the so-called minor (three-particle) singularities (see §5 of [42] and §2 of Chapter III of [46]) which weaken with increasing n. For $n \geq 4$ such singularities do not appear at all, and these kernels become Hölder functions in all their variables including the limit values $z = E \pm i0$, $E \in (\lambda_{\min}, +\infty)$. A more precise statement [42] is the following: The operator-valued functions $\mathcal{Q}^{(n)}_{\alpha\beta}(z)$ for $n \geq 4$, are of type $\mathcal{D}_{\alpha\beta}(\theta, \mu)$ for $0 < \theta < \theta_0$, $0 < \mu < 1/8$ uniformly with respect to z in any bounded set of the complex plane $\mathbb{C}$ cut along the ray $[\lambda_{\min}, +\infty)$. One can take θ, $\theta < \theta_0$, arbitrarily close to θ_0. Thus, it is convenient [42] to choose instead of $M(z)$ the new unknown $\mathcal{W}(z) = M(z) - \sum_{n=0}^{3} \mathcal{Q}^{(n)}(z)$ satisfying the equation

$$\mathcal{W}(z) \; = \; \mathcal{W}^{(0)}(z) \; - \; \mathbf{t}(z)\mathbf{R}_0(z)\Upsilon\mathcal{W}(z) \tag{4.4}$$

analogous to Eq. (2.7) but with another absolute term $\mathcal{W}^{(0)}(z) = \mathcal{Q}^{(4)}(z)$.

The solvability of Eq. (4.4) is established in the Banach space $\mathcal{B}_{\theta\mu}$ whose elements are aggregates $w = (\rho_1, \rho_2, \rho_3, \sigma_{1,1}, \ldots, \sigma_{1,n_1}, \sigma_{2,1}, \ldots, \sigma_{2,n_2}, \sigma_{3,1}, \ldots, \sigma_{3,n_3})^\dagger$ consisting of the functions $\rho_\alpha(P)$, $P \in \mathbb{R}^6$, and $\sigma_{\alpha,j}(p_\alpha)$, $p_\alpha \in \mathbb{R}^3$, $\alpha = 1\,2, 3$, $j = 1, 2, \ldots, n_\alpha$. The norm $\|w\|_{\theta\mu}$ is defined in $\mathcal{B}_{\theta\mu}$ by

$$\begin{aligned} \|w\|_{\theta\mu} \; = \; \sum_{\alpha=1}^{3} \sup\Bigg\{ & \frac{1}{\mathcal{N}(P,\theta)} \left[\rho_\alpha(P) + \frac{|\rho_\alpha(P+\Delta P) - \rho_\alpha(P)|}{|\Delta P|^\mu} \right] \\ & + \; (1+|p_\alpha|)^{2\theta} \sum_{j=1}^{n_\alpha} \left[\sigma_{\alpha,j}(p_\alpha) + \frac{\sigma_{\alpha,j}(p_\alpha + \Delta p_\alpha) - \sigma_{\alpha,j}(p_\alpha)}{|\Delta p_\alpha|^\mu} \right] \Bigg\}. \end{aligned}$$

The operator $-\mathbf{t}(z)\mathbf{R}_0(z)\Upsilon$ in (4.4) corresponds to the operator $\mathbf{A}(z)$ in $\mathcal{B}_{\theta\mu}$ the action $w' = \mathbf{A}(z)w$ of which is defined according to the representation (2.12) in such a way that

$$\begin{aligned} \rho' \; &= \; -\tilde{\mathbf{t}}\mathbf{R}_0\Upsilon(\rho + \Phi\mathbf{g}\sigma)\,, \\ \sigma' \; &= \; \Phi\mathbf{g}\Phi^*\mathbf{R}_0\Upsilon(\rho + \Phi\mathbf{g}\sigma) \end{aligned}$$

where $\mathbf{g}(z) = \operatorname{diag}\{\mathbf{g}_1(z), \mathbf{g}_2(z), \mathbf{g}_3(z)\}$, and ρ, σ and ρ', σ' stand for the components of the elements w and w', e.g. $\rho = (\rho_1, \rho_2, \rho_3)^\dagger$, $\sigma = (\sigma_{1,1}, \ldots, \sigma_{1,n_1}, \sigma_{2,1}, \ldots, \sigma_{2,n_2}, \sigma_{3,1}, \ldots, \sigma_{3,n_3})^\dagger$.

Eq. (4.4) is replaced then with the following equations in $\mathcal{B}_{\theta\mu}$

$$w_\beta(P', z) \; = \; w^{(0)}_\beta(P', z) \; + \; \mathbf{A}(z) w_\beta(P', z)\,, \qquad \beta \; = \; 1,\, 2,\, 3\,, \tag{4.5}$$

$$w_{\beta,k}(p'_\beta, z) \; = \; w^{(0)}_{\beta,k}(p'_\beta, z) \; + \; \mathbf{A}(z) w_{\beta,k}(p'_\beta, z), \quad \beta \; = \; 1,\, 2,\, 3\,, \quad k \; = \; 1,\, 2, \ldots,\, n_\beta\,, \tag{4.6}$$

where $w^{(0)}_\beta(P', z)$ stands for an element of $\mathcal{B}_{\theta\mu}$ consisting of the kernels

$$\begin{aligned} \rho^{(0)}_{\alpha\beta}(P, P', z) \; &= \; \mathcal{F}^{(0)}_{\alpha\beta}(P, P', z)\,, \qquad (\alpha \; = \; 1,\, 2,\, 3)\,, \\ \sigma^{(0)}_{\alpha,j;\beta}(p_\alpha, P', z) \; &= \; \mathcal{J}^{(0)}_{\alpha,j;\beta}(p_\alpha, P', z) \quad (\alpha \; = \; 1,\, 2,\, 3, \quad j \; = \; 1, 2, \ldots, n_\alpha)\,, \end{aligned}$$

of the components $\mathcal{F}^{(0)}(z)$ and $\mathcal{I}^{(0)}(z)$ of the function $\mathcal{W}^{(0)}(z)$ being considered at fixed β, P' and z. Analogously, $w^{(0)}_{\beta,k}(p'_\beta, z)$ is an element of $\mathcal{B}_{\theta\mu}$ consisting of the kernels

$$\begin{aligned}\rho^{(0)}_{\alpha,j;\beta}(p_\alpha, P', z) &= \mathcal{J}^{(0)}_{\alpha,j;\beta}(p_\alpha, P', z) \qquad (\alpha = 1, 2, 3),\\ \sigma^{(0)}_{\alpha,j;\beta,k}(p_\alpha, p'_\beta, z) &= \mathcal{K}^{(0)}_{\alpha,j;\beta,k}(p_\alpha, p'_\beta, z) \quad (\alpha = 1, 2, 3, \quad j = 1, 2, \ldots, n_\alpha),\end{aligned}$$

of components $\mathcal{J}^{(0)}(z)$ and $\mathcal{K}^{(0)}(z)$ of the function $\mathcal{W}^{(0)}(z)$, which are considered at fixed β, p'_β and z.

The properties of Eqs. (4.5) and (4.6) are described by Theorems 7.1 and 7.2 of the L. D. Faddeev paper [42]. We combine these theorems in the following statement.

Theorem 4.1. *For all z belonging to the complex plane $\mathbb{C}$ cut along the ray $[\lambda_{\min}, +\infty)$, the operator $\mathbf{A}(z)$ as well as all its powers $\mathbf{A}^n(z)$, $n = 2, 3, \ldots$, are defined in $\mathcal{B}_{\theta\mu'}$ for $3/2 < \theta < \theta_0$, $0 < \mu < 1/8$ on a dense subset consisting of the elements $w \in \mathcal{B}_{\theta\mu'}$, $\mu' > \mu$. If $n \geq 5$ then $\mathbf{A}^n(z)$ admits a continuation over all $\mathcal{B}_{\theta\mu}$ as a compact operator. A set of values of z where the homogeneous equation $w = \mathbf{A}(z)w$ has a nontrivial solution coincides, with the possible exception of its limit points, with the discrete spectrum $\sigma_d(H)$ of the Hamiltonian H. Therefore, the Fredholm alternative may be applied to Eqs. (4.5) and (4.6) and thereby these equations are solvable uniquely in $\mathcal{B}_{\theta\mu}$ for any $z \notin \overline{\sigma_d(H)}$ including the points $z = E \pm i0$, $E \in (\lambda_{\min}, +\infty) \setminus \overline{\sigma_d(H)}$.*

Using Theorem 4.1 one can establish as well the properties of the operator $M(z)$ itself. The corresponding statement was presented by L. D. Faddeev in [42], Theorem 5.1. In our notations it is the following:

Theorem 4.2. *Eq. (2.7) is uniquely solvable at $z \notin \overline{\sigma_d(H)}$. Its solution $M(z)$ admits the representation*

$$M(z) = \sum_{n=0}^{3} \mathcal{Q}^{(n)}(z) + \mathcal{W}(z), \tag{4.7}$$

where the operator – valued function $\mathcal{W}(z)$ is holomorphic in the variable z at $z \notin \overline{\sigma(H)}$ and its components $\mathcal{W}_{\alpha\beta}(z)$ belong to the classes $\mathcal{D}_{\alpha\beta}(\theta, \mu)$, $3/2 < \theta < \theta_0$, $0 < \mu < 1/8$, uniformly with respect to z varying in arbitrary bounded set of the complex plane $\mathbb{C}$ cut along the ray $[\lambda_{\min}, +\infty)$ and with removed neighbourhoods of the points of $\sigma_d(H)$.

4.2. Scattering matrices

Let us begin now by recalling the structure of the three – body scattering operator $\mathbf{S}$ (see e. g. the book [46], §6 of Chapter I and §3 of Chapter III). We introduce for this purpose the operator – valued function $\mathcal{T}(z)$, $\mathcal{T}(z) : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1$, for $z \in \mathbb{C} \setminus \overline{\sigma(H)}$,

$$\mathcal{T}(z) \equiv \begin{pmatrix} \Omega M(z)\Omega^\dagger & \Omega M(z)\Upsilon\Psi \\ \Psi^*\Upsilon M(z)\Omega^\dagger & \Psi^*(\Upsilon\mathbf{v} + \Upsilon M(z)\Upsilon)\Psi \end{pmatrix}. \tag{4.8}$$

In the following the notation $\mathbf{v} = \operatorname{diag}\{v_1, v_2, v_3\}$ will be used. Note that $\mathcal{T}_{00}(z) = \Omega M(z)\Omega^\dagger \equiv T(z)$ and $\mathcal{T}_{00}(z) : \mathcal{H}_0 \to \mathcal{H}_0$. The remaining components $\mathcal{T}_{01}(z) : \mathcal{H}_1 \to \mathcal{H}_0$, $\mathcal{T}_{10}(z) : \mathcal{H}_0 \to \mathcal{H}_1$ and $\mathcal{T}_{11}(z) : \mathcal{H}_1 \to \mathcal{H}_1$ are expressed by the three-body transition operators [46] (see also [57]) $U_0(z) = \Omega M(z)\Upsilon$, $U_0^\dagger = \Upsilon M(z)\Omega^\dagger$ and $U(z) = \Upsilon\mathbf{v} + \Upsilon M(z)\Upsilon$: $\mathcal{T}_{01} = U_0\Psi$, $\mathcal{T}_{10} = \Psi^* U_0^\dagger$ and $\mathcal{T}_{11} = \Psi^* U \Psi$. The operator $\mathcal{T}(z)$ is a matrix integral operator with the kernels $\mathcal{T}_{00}(P, P', z)$, $\mathcal{T}_{\alpha,i;\,0}(p_\alpha, P', z)$, $\mathcal{T}_{0;\,\beta,j}(P, p'_\beta, z)$ and $\mathcal{T}_{\alpha,i;\,\beta,j}(p_\alpha, p'_\beta, z)$ ($\alpha = 1,2,3,\ i = 1,2,\ldots,n_\alpha,\ \beta = 1,2,3,\ j = 1,2,\ldots,n_\beta$) the properties of which are determined for complex z including the limit points $z = E \pm i0$, $E > \lambda_{\min}$ by Theorem 4.2.

By $\widehat{\mathcal{T}}(z)$, $\widehat{\mathcal{T}}(z) : \widehat{\mathcal{H}}_0 \oplus \widehat{\mathcal{H}}_1 \to \widehat{\mathcal{H}}_0 \oplus \widehat{\mathcal{H}}_1$ we denote the analytic continuation in $\mathbb{C}^\pm$ (see Theorems 4.7, 4.11 and 4.12) of a matrix operator-valued function of the variable z whose components have the following kernels at $z = E \pm i0$

$$\begin{aligned}
\big(\widehat{\mathcal{T}}(E \pm i0)\big)_{00}\big(\widehat{P}, \widehat{P}'\big) &= \mathcal{T}_{00}\big(\pm\sqrt{E}\hat{P},\, \pm\sqrt{E}\hat{P}', E \pm i0\big)\,, && E > 0\,;\\
\big(\widehat{\mathcal{T}}(E \pm i0)\big)_{0;\,\beta,j}\big(\widehat{P}, \widehat{p}'_\beta\big) &= \mathcal{T}_{0;\,\beta,j}\big(\pm\sqrt{E}\widehat{P},\, \pm\sqrt{E-\lambda_{\beta,j}}\,\widehat{p}'_\beta, E \pm i0\big)\,, && E > 0\,;\\
\big(\widehat{\mathcal{T}}(E \pm i0)\big)_{\alpha,i;0}\big(\hat{p}_\alpha, \widehat{P}'\big) &= \mathcal{T}_{\alpha,i;0}\big(\pm\sqrt{E-\lambda_{\alpha,i}}\,\hat{p}_\alpha,\, \pm\sqrt{E}\widehat{P}', E \pm i0\big)\,, && E > 0\,;\\
\big(\widehat{\mathcal{T}}(E \pm i0)\big)_{\alpha,i;\,\beta,j}\big(\hat{p}_\alpha, \hat{p}'_\beta\big) &= \mathcal{T}_{\alpha,i;\,\beta,j}\big(\pm\sqrt{E-\lambda_{\alpha,i}}\,\hat{p}_\alpha,\, \pm\sqrt{E-\lambda_{\beta,j}}\,\hat{p}'_\beta, E \pm i0\big)\,,\\
&\qquad\qquad E > \max\{\lambda_{\alpha,i},\, \lambda_{\beta,j}\}\,.
\end{aligned}$$

We assume by definition that for $z = E \pm i\,0$ the product $\big(\mathbf{J}\mathcal{T}\mathbf{J}^\dagger\big)(z)$ coincides with $\widehat{\mathcal{T}}(z)$,

$$(\mathbf{J}\mathcal{T}\mathbf{J}^\dagger)(z) = \begin{pmatrix} \big(\mathrm{J}_0\mathcal{T}_{00}\mathrm{J}_0^\dagger\big)(z) & \big(\mathrm{J}_0\mathcal{T}_{01}\mathrm{J}_1^\dagger\big)(z) \\ \big(\mathrm{J}_1\mathcal{T}_{10}\mathrm{J}_0^\dagger\big)(z) & \big(\mathrm{J}_1\mathcal{T}_{11}\mathrm{J}_1^\dagger\big)(z) \end{pmatrix} \equiv \widehat{\mathcal{T}}(z)\,. \tag{4.9}$$

The elements of the matrix $(\mathbf{J}\mathcal{T}\mathbf{J}^\dagger)(z)$ are expressed in terms of amplitudes of different processes taking place in the three-body system under consideration (see Sec. 10.).

The three-body scattering operator $\mathbf{S}$ is unitary in the space $\mathcal{H}_0 \oplus \mathcal{H}_1$ and has as well as $\mathcal{T}$, a natural block structure. The kernels of its components $\mathbf{S}_{00}$, $\mathbf{S}_{0;\beta,j}$, $\mathbf{S}_{\alpha,i;0}$, $\mathbf{S}_{\alpha,i;\beta,j}$ read, repectively,

$$\mathbf{S}_{00}(P, P') = \delta(P - P') - 2\pi i\,\delta\big(P^2 - P'^2\big)\mathcal{T}_{00}\big(P, P', P'^2 + i0\big)\,, \tag{4.10}$$

$$\mathbf{S}_{0;\beta,j}\big(P, p'_\beta\big) = -2\pi i\,\delta\big(P^2 - p'^2_\beta - \lambda_{\beta,j}\big)\mathcal{T}_{0;\beta,j}\big(P, p'_\beta, \lambda_{\beta,j} + p'^2_\beta + i0\big), \tag{4.11}$$

$$\mathbf{S}_{\alpha,i;0}(p_\alpha, P') = -2\pi i\,\delta\big(\lambda_{\alpha,i} + p^2_\alpha - P'^2\big)\mathcal{T}_{\alpha,i;0}\big(p_\alpha, P', P'^2 + i0\big)\,, \tag{4.12}$$

$$\begin{aligned}\mathbf{S}_{\alpha,i;\beta,j}(p_\alpha, p'_\beta) &= \delta_{\alpha\beta}\delta_{ij}\delta\big(p_\alpha - p'_\beta\big) - 2\pi i\,\delta\big(\lambda_{\alpha,i} + p^2_\alpha - \lambda_{\beta,j} - p^2_\beta\big)\\ &\quad\times\ \mathcal{T}_{\alpha,i;\beta,j}\big(p_\alpha, p'_\beta, \lambda_{\beta,j} + p^2_\beta + i0\big)\,.\end{aligned} \tag{4.13}$$

The scattering matrices arise from $\mathbf{S}$ in the spectral decomposition of H as operators acting in the "cross section" (at fixed energy) of the space $\mathcal{H}_0 \oplus \mathcal{H}_1$ in the von Neumann direct integral [45]. As a matter of fact, the extraction of the scattering matrix from $\mathbf{S}$ corresponds to the replacements $|P|^2 \to E$, $\lambda_{\alpha,i} + p^2_\alpha \to E$ ($\alpha = 1,2,3$, $i = 1,2,\ldots,n_\alpha$)

in the expressions (4.10) – (4.13) and then to the factorization of the dependence of the kernels of $\mathbf{S}$ on the energies E and E',

$$\mathbf{S}(E,E') = -\pi i\delta(E-E')\tilde{\vartheta}(E)S'(E+i0)\tilde{\vartheta}(E') \tag{4.14}$$

where $\tilde{\vartheta}(E)$ is a diagonal matrix – function constructed from the Heaviside unit step functions $\vartheta(E)$ and $\vartheta(E-\lambda_{\beta,j})$: $\tilde{\vartheta}(E) = \mathrm{diag}\{\vartheta(E),\vartheta(E-\lambda_{1,1}),\ldots,\vartheta(E-\lambda_{1,n_1}),$ $\vartheta(E-\lambda_{2,1}),\ldots,\vartheta(E-\lambda_{2,n_2}),\vartheta(E-\lambda_{3,1}),\ldots,\vartheta(E-\lambda_{3,n_3})\}$. At $z\in\mathbb{C}$ we understand by $S'(z)$ the operator – valued function defined by $S'(z) = A^{-1}(z)\,\hat{\mathbf{I}} + \widehat{\mathcal{T}}(z)$. Hereafter, $A(z) = \mathrm{diag}\{A_0(z),A_1(z)\}$ with $A_0(z) = -\pi i z^2$ and $A_1(z) = \mathrm{diag}\left\{A^{(1)},A^{(2)},A^{(3)}\right\}$ where

$$A^{(\alpha)}(z) = \mathrm{diag}\{A_{\alpha,1}(z),\ldots,A_{\alpha,n_\alpha}(z)\} \quad \text{with} \quad A_{\alpha,j}(z) = -\pi i\sqrt{z-\lambda_{\alpha,j}}\,.$$

Continuing the factorization, $S'(z) = S(z)A^{-1}(z) = A^{-1}(z)S^{\dagger}(z)$, corresponding to separate in (4.14) the multiplier $-\pi i A^{-1}(E+i0)$ as a derivative of a measure in the von Neumann integral above [45] for $\mathcal{H}_0\oplus\mathcal{H}_1$, one arrives at the scattering matrices

$$S(z) = \hat{\mathbf{I}} + \left(\mathbf{J}\mathcal{T}\mathbf{J}^{\dagger}A\right)(z) \quad \text{and} \quad S^{\dagger}(z) = \hat{\mathbf{I}} + \left(A\,\mathbf{J}\mathcal{T}\mathbf{J}^{\dagger}\right)(z)\,. \tag{4.15}$$

In contrast to [45], it is more convenient for us to use precisely this nonsymmetrical form of the scattering matrices. The matrices $S(z)$ and $S^{\dagger}(z)$ are considered as operators in $\widehat{\mathcal{H}}_0\oplus\widehat{\mathcal{H}}_1$. At $z=E+i0$, $E>0$ these operators are unitary. At $z=E+i0$, $E<0$ there are certain truncations of $S(z)$ and $S^{\dagger}(z)$ determined by a number of open channels which are unitary in $\widehat{\mathcal{H}}_0\oplus\widehat{\mathcal{H}}_1$; namely, the matrices $\tilde{S}(E) = \hat{\mathbf{I}} + \tilde{\vartheta}(E)\big(S(E+i0)-\hat{\mathbf{I}}\big)\tilde{\vartheta}(E)$ and $\tilde{S}^{\dagger}(E) = \hat{\mathbf{I}} + \tilde{\vartheta}(E)\big(S^{\dagger}(E+i0)-\hat{\mathbf{I}}\big)\tilde{\vartheta}(E)$. It follows from Eq. (4.15) that the operator $\mathcal{T}$ may be considered as a kind of a "multichannel T – matrix" (cf. [61]) for the system of three particles.

It should be noted that the matrix $\mathcal{T}(z)$ may be replaced in Eq. (4.15) with the matrix $\mathcal{T}^{\dagger}(z)$ obtained from $\mathcal{T}(z)$ by the substitution $\Upsilon\mathbf{v}\to\mathbf{v}\Upsilon$ (respectively, $U\to U^{\dagger} = \mathbf{v}\Upsilon+\Upsilon M\Upsilon$) in the second component of the lower row of (4.8). To prove the equality $\big(\mathbf{J}\mathcal{T}^{\dagger}\mathbf{J}^{\dagger}\big)(z) = \big(\mathbf{J}\mathcal{T}\mathbf{J}^{\dagger}\big)(z)$ it suffices to observe that for $z=E\pm i0$, $E>\lambda_{\alpha,j}$ $(\alpha=1,2,3,\ j=1,2,\ldots,n_\alpha)$

$$\big(\mathrm{J}_1\Psi^*\Upsilon\mathbf{v}\Psi\mathrm{J}_1^{\dagger}\big)(z) = \big(\mathrm{J}_1\Psi^*\mathbf{v}\Upsilon\Psi\mathrm{J}_1^{\dagger}\big)(z)\,. \tag{4.16}$$

Indeed, according to Eqs. (3.12) and (3.13),

$$(\Psi^*\Upsilon\mathbf{v}\Psi)_{\alpha,i;\beta,j}(p_\alpha,p'_\beta) = -\frac{1-\delta_{\alpha\beta}}{|s_{\alpha\beta}|^3}\cdot\frac{\overline{\phi}_{\alpha,i}\big(\tilde{k}_\alpha^{(\beta)}(p_\alpha,p'_\beta)\big)\,\phi_{\beta,j}\big(\tilde{k}_\beta^{(\alpha)}(p'_\beta,p_\alpha)\big)}{\big[\tilde{k}_\alpha^{(\beta)}(p_\alpha,p'_\beta)\big]^2-\lambda_{\alpha,i}}\,, \tag{4.17}$$

$$(\Psi^*\mathbf{v}\Upsilon\Psi)_{\alpha,i;\beta,j}(p_\alpha,p'_\beta) = -\frac{1-\delta_{\beta\alpha}}{|s_{\alpha\beta}|^3}\cdot\frac{\overline{\phi}_{\alpha,i}\big(\tilde{k}_\alpha^{(\beta)}(p_\alpha,p'_\beta)\big)\,\phi_{\beta,j}\big(\tilde{k}_\beta^{(\alpha)}(p'_\beta,p_\alpha)\big)}{\big[\tilde{k}_\beta^{(\alpha)}(p'_\beta,p_\alpha)\big]^2-\lambda_{\beta,j}} \tag{4.18}$$

where

$$\tilde{k}_\gamma^{(\delta)}(q,q')) = \frac{-c_{\gamma\delta}q+q'}{s_{\gamma\delta}}\,, \quad \gamma,\,\delta = 1,\,2,\,3\,, \quad q,\,q'\in\mathbb{R}^3\,, \tag{4.19}$$

(we shall suppose later that $q, q' \in \mathbb{C}^3$). One can see easily that the denominators of the fractions (4.17) and (4.18) coincide on the energy shells $|p_\alpha| = \sqrt{E-\lambda_{\alpha,i}}$, $|p'_\beta| = \sqrt{E-\lambda_{\beta,j}}$, $E > \lambda_{\alpha,i}$, $E > \lambda_{\beta,j}$:

$$
\begin{aligned}
\big(\tilde{k}_\alpha^{(\beta)}\big)^2 - \lambda_{\alpha,i} &= \big(\tilde{k}_\beta^{(\alpha)}\big)^2 - \lambda_{\beta,j} \\
&= \frac{1}{|s_{\alpha\beta}|^2}\big(E-\lambda_{\alpha,i}+E-\lambda_{\beta,j} \\
&\qquad - 2c_{\alpha\beta}\sqrt{E-\lambda_{\alpha,i}}\sqrt{E-\lambda_{\beta,j}}\,(\hat{p}_\alpha,\hat{p}'_\beta) - s^2_{\alpha\beta}E\big)\,.
\end{aligned} \tag{4.20}
$$

Meanwhile, the expression (4.20) cannot become zero at $E > \lambda_{\alpha,i}$, $E > \lambda_{\beta,j}$ (see Lemma 4.4). It follows now from Eqs. (4.17), (4.18) and (4.20) that the equality (4.16) is true.

Along with $S(z)$ and $S^\dagger(z)$ we shall consider further also the truncated scattering matrices

$$
S_l(z) \equiv \hat{\mathbf{I}} + \big(\tilde{L}\mathbf{J}\mathcal{T}\mathbf{J}^\dagger L A\big)(z) \quad \text{and} \quad S_l^\dagger(z) \equiv \hat{\mathbf{I}} + \big(AL\mathbf{J}\mathcal{T}\mathbf{J}^\dagger \tilde{L}\big)(z)\,, \tag{4.21}
$$

where the multi−index

$$
l = (l_0, l_{1,1}, \ldots, l_{1,n_1}, l_{2,1}, \ldots, l_{2,n_2}, l_{3,1}, \ldots, l_{3,n_3}) \tag{4.22}
$$

has components $l_0 = 0$ or $l_0 = \pm 1$ and $l_{\alpha,j} = 0$ or $l_{\alpha,j} = 1$, $\alpha = 1,2,3$, $j = 1,2,\ldots,n_\alpha$. By L and $\tilde{L}$ in (4.21) and in the following we mean the diagonal number matrices

$$
L = \operatorname{diag}\{l_0, l_{1,1}, \ldots, l_{1,n_1}, l_{2,1}, \ldots, l_{2,n_2}, l_{3,1}, \ldots, l_{3,n_3}\} \tag{4.23}
$$

and

$$
\tilde{L} = \operatorname{diag}\{|l_0|, l_{1,1}, \ldots, l_{1,n_1}, l_{2,1}, \ldots, l_{2,n_2}, l_{3,1}, \ldots, l_{3,n_3}\}\,, \tag{4.24}
$$

corresponding to the multi−index l. The matrix $\tilde{L}$ is evidently a projection in $\widehat{\mathcal{H}}_0 \oplus \widehat{\mathcal{H}}_1$ on the subspace $\widehat{\mathcal{H}}_1^{(l)}$ if $l_0 = 0$ or on the subspace $\widehat{\mathcal{H}}_0 \oplus \widehat{\mathcal{H}}_1^{(l)}$ if $l_0 \neq 0$. Here $\widehat{\mathcal{H}}_1^{(l)} = \bigoplus_{l_{\alpha,j}\neq 0} \widehat{\mathcal{H}}^{(\alpha,j)}$ in both cases.

As can be seen from formulas (4.15) and (4.8) the scattering matrices $S(z)$ and $S^\dagger(z)$ include the kernels $M_{\alpha\beta}(P,P',z)$ taken on the energy shells: their arguments $P \in \mathbb{R}^6$ and $P' \in \mathbb{R}^6$ are connected with the energy $z = E + i0$ by Eq. (2.15) at $E > 0$ or Eqs. (2.16) at $E > \lambda_{\alpha,j}$. We establish below [see formula (7.34)] that the analytic continuation of the matrix $M(z)$ into unphysical sheets of the energy z is expressed in terms of the analytic continuation of the truncated scattering matrices $S_l(z)$ or $S_l^\dagger(z)$ and half−on−shell Faddeev components $M_{\alpha\beta}(z)$ taken in the physical sheet. More precisely, along with $S_l(z)$, the final formula (7.34) includes the matrices $\big(L_0\mathbf{J}_0 M\big)(z)$, $\big(L_1 \mathrm{J}_1 \Psi^* \Upsilon M\big)(z)$ and $\big(M\mathbf{J}_0^\dagger L_0\big)(z)$, $\big(M\Upsilon\Psi \mathrm{J}_1^\dagger L_1\big)(z)$. Here, l is the multi−index (4.22) and $L = \operatorname{diag}\{L_0, L_1\}$, the respective matrix (4.23) with $L_0 = l_0$ and $L_1 = \operatorname{diag}\{l_{1,1}, \ldots, l_{1,n_1}, l_{2,1}, \ldots, l_{2,n_2}, l_{3,1}, \ldots, l_{3,n_3}\}$.

Further, we formulate some statements (Theorems 4.7 – 4.12) concerning the existence of an analytic continuation of the above matrices and their domains of holomorphy. Proofs of these statements will be based on analysis [42] of the Faddeev

equations (2.7). For this, one must pay a special attention to the study of the domains of holomorphy in z of the functions

$$\left[p_\alpha^2 + {p'_\beta}^2 - 2c_{\alpha\beta}(p_\alpha, p'_\beta) - s_{\alpha\beta}^2 z\right]^{-1} \tag{4.25}$$

with one or both arguments p_α and p'_β situated on the energy shells (2.15) or (2.16). The functions (4.25) arise when iterating Eq. (2.7) because of the presence of the multiplier $\mathbf{R}_0$ in the operator $-\mathbf{t}\mathbf{R}_0\Upsilon$. Also, the functions (4.25) display the singularities (3.13) of the eigenfunctions $\psi_{\alpha,j}$, $\alpha = 1, 2, 3,\ j = 1, 2, \ldots, n_\alpha$.

In the case where the arguments p_α and/or p'_β are taken on the shells (2.16), $p_\alpha = \sqrt{z - \lambda_{\alpha,i}}\,\hat{p}_\alpha$ and $p'_\beta = \sqrt{z - \lambda_{\beta,j}}\,\hat{p}'_\alpha$, the holomorphy domains of the functions (4.25) with respect to the variable z are described by the following simple lemmas.

Lemma 4.3. *For any $\rho \geq 0$, $-1 \leq \eta \leq 1$, the domain*

$$\operatorname{Re} z \;>\; \frac{\lambda}{c^2} \;+\; \frac{c^2}{4s^2|\lambda|}\,(\operatorname{Im} z)^2 \tag{4.26}$$

contains no roots z of the equation

$$z \;-\; \lambda \;+\; \rho \;+\; 2c\sqrt{z-\lambda}\,\sqrt{\rho}\,\eta \;-\; s^2 z \;=\; 0\,, \tag{4.27}$$

with $\lambda < 0$, $0 < |c| < 1$ and $s^2 = 1 - c^2$. For any number $z \in \mathbb{C}$ outside of the domain (4.26) one can always find values of the parameters $\rho \geq 0$ and η, $-1 \leq \eta \leq 1$, such that the left – hand side of Eq. (4.27) becomes equal to zero at the point z.

Lemma 4.4. *Consider the equation*

$$z \;-\; \lambda_1 \;+\; z \;-\; \lambda_2 \;+\; 2c\,\sqrt{z-\lambda_1}\,\sqrt{z-\lambda_2}\,\eta \;-\; s^2 z \;=\; 0 \tag{4.28}$$

where $\eta \in [-1, 1]$, $\lambda_1 \leq \lambda_2 < 0$, $0 < c < 1$ and $s^2 = 1 - c^2$. Then the following assertions hold:

1) *If $|\lambda_2| > c^2|\lambda_1|$, then for all $\eta \in [-1, 1]$ Eq. (4.28) has a unique root z and this root is real. Moreover, $z = z_+$ if $\eta \geq 0$, and $z = z_-$ if $\eta \leq 0$ with*

$$z_\pm \;=\; \frac{\left(1 + c^2 - 2c^2\eta^2\right)(\lambda_1 + \lambda_2) \pm 2\sqrt{c^2\eta^2[\lambda_1\lambda_2\, s^4 - (\lambda_2 - \lambda_1)^2 c^2(1 - \eta^2)]}}{(1 + c^2)^2 - 4c^2\eta^2}\,. \tag{4.29}$$

If η runs through the interval $[-1, 1]$, the roots $z_\pm$ fill the interval $[z_{\rm lt}, z_{\rm rt}]$ whose ends are

$$z_{\rm lt} \;=\; \frac{1}{s^2}\left[\,-|\lambda_1| - |\lambda_2| - 2c\sqrt{|\lambda_1|\cdot|\lambda_2|}\,\right] \tag{4.30}$$

and

$$z_{\rm rt} \;=\; \frac{1}{s^2}\left[\,-|\lambda_1| - |\lambda_2| + 2c\sqrt{|\lambda_1|\cdot|\lambda_2|}\,\right], \quad z_{\rm rt} \;<\; \lambda_1\,. \tag{4.31}$$

2) *If $|\lambda_2| = c^2|\lambda_1|$, then Eq. (4.28) has two real roots:*
   a) *the root $z = \lambda_1$ existing for all $\eta \in [-1, 1]$;*
   b) *the root $z = z_-$ given by (4.29) which exists for $-1 \leq \eta \leq 0$ only.*

For $z_{\mathrm{lt}} = -|\lambda_1|\left(1+2c^4/s^2\right)$, $-1 \le \eta \le 1$ *these roots together fill the interval* $[z_{\mathrm{lt}}, \lambda_1]$.

3) *If* $|\lambda_2| < c^2|\lambda_1|$, *then*

a) *for* $-1 \le \eta \le \eta^*$, $\eta^* = (\sqrt{c^2-\rho}\sqrt{1-c^2\rho})/(c(1-\rho))$ *and* $\rho = |\lambda_2|/|\lambda_1|$, *Eq.* (4.28) *has two real roots* $z_\pm$ *given by* (4.29) *which fill the interval* $[z_{\mathrm{lt}}, z_{\mathrm{rt}}]$ *with ends* (4.30) *and* (4.31), $z_{\mathrm{rt}} < \lambda_1$;

b) *for* $\eta^* < \eta \le 0$ *Eq.* (4.28) *has two complex roots* $z_\pm$ *described again by Eq.* (4.29). *If* η *varies, these roots fill an ellipse centered at the point*

$$z_c \;=\; -|\lambda_1|\left[1+\frac{(c^2-\rho)^2}{s^2(1+c^2)(1+\rho)}\right].$$

The half–axes of this ellipse are given by

$$a \;=\; |\lambda_1|\cdot\frac{(c^2-\rho)(1-c^2\rho)}{(1+c^2)s^2(1+\rho)}$$

(*along the real axis*) *and*

$$b \;=\; |\lambda_1|\cdot\frac{(c^2-\rho)(1-c^2\rho)}{(1+c^2)s^2(1-\rho)\sqrt{(1+c^2)^2-4c^2\eta^{*2}}}$$

(*along the imaginary axis*). *The right vertex of the ellipse is located at the point*

$$z_{\mathrm{rt}}^{(\mathrm{e})} \;=\; z_c + a \;=\; -\frac{|\lambda_1|+|\lambda_2|}{1+c^2}$$

situated between λ_1 *and* λ_2. *Its left vertex is*

$$z_{\mathrm{lt}}^{(\mathrm{e})} \;=\; z_c - a \;<\; z_{\mathrm{rt}}\,.$$

Lemma 4.5. *Let the parameters of the equation*

$$z \,-\, \lambda \,+\, z\nu \,+\, 2c\sqrt{z}\,\sqrt{z-\lambda}\,\sqrt{\nu}\eta \,-\, s^2 z \;=\; 0 \tag{4.32}$$

satisfy the conditions $\nu \in [0,1]$, $\eta \in [-1,1]$, $\lambda < 0$, $c \in (0,1)$ *and* $s^2 = 1-c^2$. *Then, if* ν *and* η *run through the above ranges, the roots* z *of Eq.* (4.32) *fill the ray* $\left(-\infty, \lambda/(1+c^4)\right]$ *and the circle centered at the point* $z_c = \lambda/(1-c^4)$ *the radius of which is equal to* $c^2\lambda/(1-c^4)$.

Also, we shall use

Lemma 4.6. *Let the parameters* ν' *and* η *of the equation*

$$\rho \,+\, z\nu' \,+\, 2c\sqrt{z}\,\sqrt{\nu'}\sqrt{\rho}\eta \,-\, s^2 z \;=\; 0$$

run through the intervals $0 \le \nu' \le 1$ *and* $-1 \le \eta \le 1$, *respectively, and* $c > 0$, $s^2 = 1-c^2$, $z \in \mathbb{C}$ *be fixed. Then the roots* ρ *of this equation fill a set consisting of the line segment* $[0,z]$ *in the complex plane* $\mathbb{C}$ *and a circle centered at the origin, the radius of which is equal to* $c^2|z|$.

4.3. Analytic continuation of the matrices $M\Upsilon\Psi \mathrm{J}_1^\dagger$, $\mathrm{J}_1\Psi^*\Upsilon M$ and $\mathcal{T}_{11}$

Let $\Pi_b^{(\beta,j)}$ be a domain in the complex plane $\mathbb{C}$ cut along the ray $[\lambda_{\min},+\infty)$ where the conditions (4.26) with $\lambda=\lambda_{\beta,j}$, $c=c_{\alpha\beta}$ and the inequalities

$$\operatorname{Re} z \;>\; \lambda_{\beta,j} \;-\; s_{\alpha\beta}^2 b^2 \;+\; \frac{1}{4s_{\alpha\beta}^2 b^2}\,(\operatorname{Im} z)^2 \tag{4.33}$$

are valid simultaneously for all $\alpha=1,2,3$, $\alpha\neq\beta$. In the case of the potentials (2.2) one has to take $b=+\infty$ in (4.33).

By $\mathcal{R}_{\alpha,i;\,\beta,j}$, $\alpha\neq\beta$ we denote a domain complementary in $\mathbb{C}\setminus[\lambda_{\min},+\infty)$ to the set filled by the roots of Eq. (4.28) in the case where $\lambda_1=\min\{\lambda_{\alpha,i},\lambda_{\beta,j}\}$, $\lambda_2=$ $=\max\{\lambda_{\alpha,i},\lambda_{\beta,j}\}$, $c=|c_{\alpha\beta}|$ and $\eta=\big(\hat p_\alpha,\hat p'_\beta\big)$ runs through the interval $[-1,1]$.

Theorem 4.7. *The matrix integral operator $L'_1\hat{\mathcal{T}}_{11}(z)L''_1$, $z=E\pm i0$ acting in $\widehat{\mathcal{H}}_1$ admits analytic continuation in z from the edges of the ray $E\in(\lambda,+\infty)$,*

$$\lambda \;=\; \max_{\substack{l'_{\gamma,k}\neq 0,\\ l''_{\gamma,k}\neq 0}} \lambda_{\gamma,k}\,,$$

on the domain

$$\Pi^{(\mathrm{hol})}_{l'l''} \;=\; \left[\;\bigcap_{\substack{l'_{\alpha,i}\neq 0\\ l''_{\beta,j}\neq 0}} \mathcal{R}_{\alpha,i;\,\beta,j}\right]\bigcap\left[\;\bigcap_{\substack{l'_{\gamma,k}\neq 0,\\ l''_{\gamma,k}\neq 0}} \Pi_b^{(\gamma,k)}\right]\Big\backslash\,\overline{\sigma(H)} \tag{4.34}$$

where

$$\begin{aligned} l'_1 \;&=\; \operatorname{diag}\big(l'_0, l'_{1,1},\,\ldots, l'_{1,n_1}, l'_{2,1},\,\ldots, l'_{2,n_2}, l'_{3,1},\,\ldots, l'_{3,n_3}\big)\,,\\ l''_1 \;&=\; \operatorname{diag}\big(l''_0, l''_{1,1},\,\ldots, l''_{1,n_1}, l''_{2,1},\,\ldots, l''_{2,n_2}, l''_{3,1},\,\ldots, l''_{3,n_3}\big) \end{aligned}$$

with $l'_0=l''_0=0$. The nontrivial kernels $\big(L'_1\hat{\mathcal{T}}_{11}(z)L''_1\big)_{\alpha,i;\,\beta,j}(\hat p_\alpha,\hat p'_\beta,z)$, $l'_{\alpha,i}\neq 0$, $l''_{\beta,j}\neq 0$ turn out to be functions holomorphic in $z\in\Pi^{(\mathrm{hol})}_{l'l''}$ and real – analytic with respect to $\hat p_\alpha,\hat p'_\beta\in S^2$.

Remark 4.8. The domains $\Pi^{(\mathrm{hol})}_{l'l''}$ and $\Pi^{(\mathrm{hol})}_{l''l'}$ coincide, $\Pi^{(\mathrm{hol})}_{l'l''}=\Pi^{(\mathrm{hol})}_{l''l'}$.
If $l'=l''=l$, we use for $\Pi^{(\mathrm{hol})}_{l'l''}$ the notation $\Pi^{(\mathrm{hol})}_{l}$, and we have

$$\Pi^{(\mathrm{hol})}_{l} \;=\; \Pi^{(\mathrm{hol})}_{ll}\,. \tag{4.35}$$

Theorem 4.9. *Let $L_0=l_0=0$. Then the matrices $\big(M\Upsilon\Psi\mathrm{J}_1^\dagger L_1\big)(z)$ and $\big(L_1\mathrm{J}_1\Psi^*\Upsilon M\big)(z)$, $z=E\pm i0$ admit analytic continuation in z from the edges of the ray $E\in(\lambda,+\infty)$,*

$$\lambda \;=\; \max_{(\beta,j):\,\lambda_{\beta,j\neq 0}} \lambda_{\beta,j}$$

to the domain $\Pi_l^{(\mathrm{hol})} \setminus \overline{\sigma(H)}$ *as bounded for* $z \notin [\lambda_{\min}, +\infty)$ *operator – valued functions of the variable* z,

$$\big(M\Upsilon\Psi \mathrm{J}_1^\dagger L_1\big)(z) \;:\; \widehat{\mathcal{H}}_1 \;\longrightarrow\; \mathcal{G}_0 \quad \textit{and} \quad \big(L_1 \mathrm{J}_1 \Psi^* \Upsilon M\big)(z) \;:\; \mathcal{G}_0 \;\longrightarrow\; \widehat{\mathcal{H}}_1 \,.$$

First we prove Theorem 4.9, then Theorem 4.7.

P r o o f of Theorem 4.9. We give the proof for the case of the matrix $M\Upsilon\Psi\mathrm{J}_1^\dagger L_1$. It follows from Theorem 4.1 that the kernels

$$\big(M\Upsilon\Psi \mathrm{J}_1^\dagger\big)_{\alpha;\beta,j}(P, \hat{p}'_\beta, E \pm i0) \;\equiv\; \sum_{\gamma \neq \beta} \int_{\mathbb{R}^3} dk'_\beta \, M_{\alpha\gamma}(P, P', E \pm i0)\, \psi_{\beta,j}\big(k'_\beta\big)\,,$$
$$P' \;=\; \big(k'_\beta, \pm\sqrt{E - \lambda_{\beta,j}}\, \hat{p}'_\beta\big)$$

of the nontrivial elements $\big(M\Upsilon\Psi\mathrm{J}_1^\dagger\big)_{\alpha;\beta,j}(E \pm i0)$, $\alpha = 1,2,3$, $l_{\beta,j} \neq 0$ are defined at $E > \lambda_{\beta,j}$. As a whole, the matrix $\big(M\Upsilon\Psi\mathrm{J}^\dagger L_1\big)(z)$ at $z = E \pm i0$,

$$E \;>\; \lambda \;=\; \max_{(\beta,j):\, l_{\beta,j} \neq 0} \lambda_{\beta,j}$$

satisfies the Faddeev equation(s)

$$\big(M\Upsilon\Psi\mathrm{J}_1^\dagger L_1\big)(z) \;=\; \big(\mathbf{t}\Upsilon\Psi\mathrm{J}_1^\dagger L_1\big)(z) \;-\; \big(\mathbf{t}\mathbf{R}_0\Upsilon M\Upsilon\Psi\mathrm{J}_1^\dagger L_1\big)(z)\,, \tag{4.36}$$

the absolute term $\mathbf{t}\Upsilon\Psi\mathrm{J}_1^\dagger L_1$ of which at $l_{\beta,j} \neq 0$ has the kernels

$$\begin{aligned} &\big(\mathbf{t}\Upsilon\Psi\mathrm{J}_1^\dagger\big)_{\alpha;\beta,j}(P, \hat{p}'_\beta, z) \\ &= \; (1 - \delta_{\alpha\beta}) \cdot \frac{1}{|s_{\alpha\beta}|^3} \cdot t_\alpha\big(k_\alpha, \tilde{k}_\alpha^{(\beta)}\big(p_\alpha, \sqrt{z - \lambda_{\beta,j}}\, \hat{p}'_\beta\big), z - p_\alpha^2\big) \\ &\quad \times \; \psi_{\beta,j}\big(\tilde{k}_\beta^{(\alpha)}\big(\sqrt{z - \lambda_{\beta,j}}\, \hat{p}'_\beta\big), p_\alpha\big)\big)\,. \end{aligned} \tag{4.37}$$

Evidently, these admit analytic continuation in z in the usual sense as smooth (even real – analytic) functions of the arguments $P \in \mathbb{R}^6$ and $\hat{p}'_\beta \in S^2$ on the domains where the respective two – body eigenfunctions

$$\psi_{\beta,j}\Big(\tilde{k}_\beta^{(\alpha)}\Big) = -\,\frac{\phi_{\beta,j}\Big(\tilde{k}_\beta^{(\alpha)}\Big)}{\tilde{k}_\beta^{(\alpha)2} - \lambda_{\beta,j}}$$

are regular. The condition of regularity of the functions $\psi_{\beta,j}\big(\tilde{k}_\beta^{(\alpha)}\big)$ is equivalent to the requirements

$$\Big[\tilde{k}_\beta^{(\alpha)}\big(\sqrt{z - \lambda_{\beta,j}}\, \hat{p}'_\beta,\, p_\alpha\big)\Big]^2 \;\neq\; \lambda_{\beta,j} \quad \text{for all} \quad p_\alpha \in \mathbb{R}^3 \;\text{ and }\; \hat{p}'_\beta \in S^2. \tag{4.38}$$

As follows from Eqs. (4.19), the requirements contrary to the conditions (4.38), are equivalent to the conditions (4.27) with $\lambda = \lambda_{\beta,j}$, $\mu = |p_\alpha|^2$, $c = |c_{\beta\alpha}|$, $s = |s_{\beta\alpha}|$ and

$\eta = (\hat{p}_\alpha, \hat{p}'_\beta)$. Thereby, on the basis of Lemma 4.3, we conclude that the inequalities (4.38) at $l_{\beta,j} \neq 0$ are satisfied only if the inequalities (4.26) with $\lambda = \lambda_{\beta,j}$, $c = c_{\alpha\beta}$ are valid.

In the case of the potentials (2.3), one has to require additionally the variables $\tilde{k}_\alpha^{(\beta)}$ and $\tilde{k}_\beta^{(\alpha)}$ to belong to the analyticity strips W_b and W_{2b} of the kernels $t_\alpha\big(k_\alpha, \tilde{k}_\alpha^{(\beta)}, z - p_\alpha^2\big)$ and form factors $\phi_{\beta,j}\big(\tilde{k}_\beta^{(\alpha)}\big)$, respectively, i. e.,

$$(4.39) \qquad \left|\mathrm{Im}\, \tilde{k}_\alpha^{(\beta)}\big(p_\alpha, \sqrt{z-\lambda_{\beta,j}}\, \hat{p}'_\beta\big)\right| < b\,, \qquad \left|\mathrm{Im}\, \tilde{k}_\beta^{(\alpha)}\big(\sqrt{z-\lambda_{\beta,j}}\, \hat{p}'_\beta, p_\alpha\big)\right| < 2b\,.$$

At $p_\alpha \in \mathbb{R}^3$, $\hat{p}'_\beta \in S^2$, the inequalities (4.39) are certainly satisfied if the conditions (4.33) are satisfied.

Let $\big(M\Upsilon\Psi \mathrm{J}_1^\dagger L_1\big)_{\beta,j}(z)$ be the (β, j) – th column of the matrix $\big(M\Upsilon\Psi \mathrm{J}_1^\dagger L_1\big)(z)$, $l_{\beta,j} \neq 0$. The kernels $\big(M\Upsilon\Psi \mathrm{J}_1^\dagger L_1\big)(z)_{\alpha;\,\beta,j}(P, \hat{p}'_\beta, z)$, $\alpha = 1, 2, 3$ of its components at fixed $\hat{p}_\beta$ satisfy a respective system of the Faddeev equations following from (4.36). As we established, the absolute terms (4.37) of this system admit analytic continuation in z from the edges of the ray $(\lambda_{\beta,j}, +\infty)$ over the whole domain $\Pi_b^{(\beta,j)}$. The same may be stated also regarding the iterations $\mathcal{Q}_{\beta,j}^{(n)}(z) = \big(-\mathbf{t}(z)\mathbf{R}_0(z)\Upsilon\big)^n \big(\mathbf{t}\Upsilon\Psi \mathrm{J}_1^\dagger L_1\big)_{\beta,j}(z)$. Embedding the Faddeev equations for the column $\big(M\Upsilon\Psi \mathrm{J}_1^\dagger L_1\big)_{\beta,j}(z)$ into the Banach space $\mathcal{B}_{\theta\mu'}$ in the same way as it was done for Eq. (2.7), one finds that the kernels $\big(M\Upsilon\Psi \mathrm{J}_1^\dagger L_1\big)_{\alpha;\,\beta,j}(P, \hat{p}'_\alpha, z)$ may be continued on $\Pi_b^{(\beta,j)} \setminus \overline{\sigma(H)}$ as analytic functions of the variable z for all $P \in \mathbf{R}^6$, $\hat{p}'_\beta \in S^2$. It follows from the estimates of the rate of decrease of these kernels that they represent the operator $\big(M\Upsilon\Psi \mathrm{J}_1^\dagger L_1\big)_{\beta,j}(z)$: $\widehat{\mathcal{H}}^{(\beta,j)} \to \mathcal{G}_0$, bounded for $z \notin [\lambda_{\min}, +\infty) \cup \overline{\sigma_d(H)}$ and depending analytically on $z \in \Pi_b^{(\beta,j)} \setminus \overline{\sigma(H)}$. Therefore, we have proved the statement of the theorem for the matrix $\big(M\Upsilon\Psi \mathrm{J}_1^\dagger L_1\big)(z)$. The statement for $\big(L_1 \mathrm{J}_1 \Psi^* \Upsilon M\big)(z)$ can be established quite similarly.

The proof of Theorem 4.9 is completed. □

P r o o f of Theorem 4.7. Note first that when proving Theorem 4.9 we found out in passing that the kernels $\big(\mathcal{Q}^{(n)}\Upsilon\Psi \mathrm{J}_1^\dagger\big)_{\alpha;\,\beta,j}(P, \hat{p}_\beta, E \pm i0)$, $E > \lambda_{\beta,j}$, of the operators $\mathcal{Q}^{(n)}\Upsilon\Psi \mathrm{J}_1^\dagger$ admit already for $n = 0$ an immediate analytic continuation on the domain of $z \in \Pi_b^{(\beta,j)}$ as real – analytic functions of the variables $P \in \mathbb{R}^6$ and $\hat{p}'_\beta \in S^2$. The same may be stated as well for the kernels $(\mathrm{J}_1\Psi^*\Upsilon\mathcal{Q}^{(n)})_{\beta,j;\,\alpha}(\hat{p}_\beta, P', E \pm i0)$, $E > \lambda_{\beta,j}$, of the operators $\mathrm{J}_1\Psi^*\Upsilon\mathcal{Q}^{(n)}$ when continuing them in $\Pi_b^{(\beta,j)}$.

On the basis of this note one may consider the relation

$$(4.40) \qquad \begin{aligned} L'_1 \widehat{\mathcal{T}}_{11} L''_1 \;&=\; L'_1 \mathrm{J}_1 \Psi^* \Upsilon \mathbf{v} \Psi \mathrm{J}_1^\dagger L''_1 \\ &\quad+\; L'_1 \mathrm{J}_1 \Psi^* \Upsilon \left(\mathbf{t} - \mathbf{t}\mathbf{R}_0\Upsilon\mathbf{t} + \mathbf{t}\mathbf{R}_0\Upsilon M \Upsilon \mathbf{R}_0 \mathbf{t}\right) \Upsilon \Psi \mathrm{J}_1^\dagger L''_1\,, \end{aligned}$$

following from Eq. (2.9) at $m = n = 0$ and

$$\begin{aligned} L'_1 \;&=\; \mathrm{diag}\left\{l'_{1,1}, \ldots, l'_{1,n_1}, l'_{2,1}, \ldots, l'_{2,n_2}, l'_{3,1}, \ldots, l'_{3,n_3}\right\}, \\ L''_1 \;&=\; \mathrm{diag}\left\{l''_{1,1}, \ldots, l''_{1,n_1}, l''_{2,1}, \ldots, l''_{2,n_2}, l''_{3,1}, \ldots, l''_{3,n_3}\right\}, \end{aligned}$$

as a representation for analytic continuation of the matrix $L_1'\widehat{\mathcal{T}}_{11}L_1'' : \widehat{\mathcal{H}_1} \to \widehat{\mathcal{H}_1}$ in terms of the operator $M(z)$ with the kernels $M_{\alpha\beta}(P, P', z)$, the arguments P and P' of which are real, $P, P' \in \mathbb{R}^6$. Additionally, one knows already that the summand $L_1'\mathrm{J}_1\Psi^*\Upsilon\mathcal{Q}\Upsilon\Psi\mathrm{J}_1^\dagger L_1''$ with $\mathcal{Q} = \mathbf{t}\mathbf{R}_0\Upsilon M\Upsilon\mathbf{R}_0\mathbf{t}$ admits analytic continuation in z on the domain

$$\bigcap_{\substack{l'_{\beta,j} \neq 0,\\ l''_{\beta,j} \neq 0}} \Pi_b^{(\beta,j)} \setminus \overline{\sigma(H)}$$

as a matrix integral operator in $\widehat{\mathcal{H}_1}$, since the operators $L_1'\mathrm{J}_1\Psi^*\Upsilon\mathbf{t}$ and $\mathbf{t}\Upsilon\Psi\mathrm{J}_1^\dagger L_1''$ may be continued on this domain. Now, one only has to find a domain of holomorphy for the rest of the summands in (4.41).

The nontrivial kernels

$$(4.41)\quad \left(L_1'\mathrm{J}_1\Psi^*\Upsilon\mathbf{v}\Psi\mathrm{J}_1^\dagger L_1''\right)_{\alpha,i;\,\beta,j}(\hat{p}_\alpha, \hat{p}_\beta, z)\,, \quad l'_{\alpha,i} \neq 0\,,\ l''_{\beta,j} \neq 0\,,\ \hat{p}_\alpha \in S^2,\ \hat{p}'_\beta \in S^2,$$

of the first summand in (4.41) look like (4.17) where one has to take $p_\alpha = \sqrt{z - \lambda_{\alpha,i}}\,\hat{p}_\alpha$, $p'_\beta = \sqrt{z - \lambda_{\beta,j}}\,\hat{p}'_\beta$. Furthermore, the functions $\overline{\phi}_{\alpha,i}\big(\tilde{k}_\alpha^{(\beta)}\big)$ have to be replaced with their analytic continuations $\tilde{\phi}_{\alpha,i}\big(\tilde{k}_\alpha^{(\beta)}\big)$ in a domain of complex $\tilde{k}_\alpha^{(\beta)}$. It is clear that the kernels (4.41) are holomorphic in that domain of the variable z where their denominators (4.20) can be equal to zero for no $\hat{p}_\alpha, \hat{p}'_\beta \in S^2$. This domain is described by Lemma 4.4. It follows from Eq. (4.17) in accordance with this lemma that the kernel (4.41) is a holomorphic function of z in the domain $\mathcal{R}_{\alpha,i;\,\beta,j}$.

In the case of the potentials (2.3) we require additionally the arguments $\tilde{k}_\alpha^{(\beta)}(p_\alpha, p'_\beta)$ and $\tilde{k}_\beta^{(\alpha)}(p'_\beta, p_\alpha)$ of the formfactors $\tilde{\phi}_{\alpha,i}$ and $\phi_{\beta,j}$ to belong at $p_\alpha = \sqrt{z - \lambda_{\alpha,i}}\,\hat{p}_\alpha$ and $p'_\beta = \sqrt{z - \lambda_{\beta,j}}\,\hat{p}'_\beta$ to the holomorphy strips W_{2b}. Note meanwhile that if $\lambda_1 < \lambda_2$ then $\operatorname{Im}\sqrt{z - \lambda_1} \leq \operatorname{Im}\sqrt{z - \lambda_2}$. Thus, the conditions $\big|\operatorname{Im}\tilde{k}_\alpha^{(\beta)}\big| < 2b$, $\big|\operatorname{Im}\tilde{k}_\beta^{(\alpha)}\big| < 2b$ are satisfied at

$$(4.42)\qquad \operatorname{Re} z \;>\; \lambda - \frac{4|s_{\alpha\beta}|^2}{(1 + |c_{\alpha\beta}|)^2}\, b^2 \;+\; \frac{(1 + |c_{\alpha\beta}|)^2}{16|s_{\alpha\beta}|^2\, b^2}\,(\operatorname{Im} z)^2$$

where $\lambda = \max\{\lambda_{\alpha,i}, \lambda_{\beta,j}\}$. Since $1 + |c_{\alpha\beta}| < 2$, the inequalities (4.42) are obeyed automatically if

$$z \in \bigcap_{\substack{l'_{\gamma,k} \neq 0,\\ l''_{\gamma,k} \neq 0}} \Pi_b^{(\gamma,k)}\,.$$

The two remaining terms, $L_1'\mathrm{J}_1\Psi^*\Upsilon\mathbf{t}\Upsilon\Psi\mathrm{J}_1^\dagger L_1''$ and $L_1'\mathrm{J}_1\Psi^*\Upsilon\mathbf{t}\mathbf{R}_0\Upsilon\mathbf{t}\Upsilon\Psi\mathrm{J}_1^\dagger L_1''$, have respectively the kernels

$$(4.43)\qquad \begin{aligned} &\sum_{\gamma\neq\alpha,\beta} \frac{1}{|s_{\alpha\gamma}s_{\beta\gamma}|^3} \int_{\mathbb{R}^3} dq\, \tilde{\psi}_{\alpha,i}\big(\tilde{k}_\alpha^{(\gamma)}(p_\alpha, q)\big)\, \psi_{\beta,j}\big(\tilde{k}_\beta^{(\gamma)}(p'_\beta, q)\big) \\ &\qquad\qquad \times\; t_\gamma\big(\tilde{k}_\alpha^{(\alpha)}(q, p_\alpha), \tilde{k}_\gamma^{(\beta)}(q, p'_\beta), z - q^2\big) \end{aligned}$$

and

$$\sum_{\substack{\gamma\neq\alpha,\ \delta\neq\beta\\ \gamma\neq\delta}} \frac{1}{|s_{\alpha\gamma}|^3|s_{\gamma\delta}||s_{\beta\delta}|^3}\cdot\int_{\mathbb{R}^3}dq\int_{\mathbb{R}^3}dq'\,\tilde{\psi}_{\alpha,i}\big(\tilde{k}_\alpha^{(\gamma)}(p_\alpha,q)\big)\,\psi_{\beta,j}\big(\tilde{k}_\beta^{(\delta)}(p'_\beta,q)\big)$$

$$\times\ \frac{t_\gamma\big(\tilde{k}_\alpha^{(\alpha)}(q,p_\alpha),\tilde{k}_\gamma^{(\delta)}(q,q'),z-q^2\big)\cdot t_\delta\big(\tilde{k}_\delta^{(\gamma)}(q,q'),\tilde{k}_\gamma^{(\delta)}(q',p'_\beta),z-{q'}^2\big)}{q^2+{q'}^2-2c_{\gamma\delta}(q,q')-s_{\gamma\delta}^2\,z}\,. \tag{4.44}$$

Here one has to take $p_\alpha=\sqrt{z-\lambda_{\alpha,i}}\,\hat{p}_\alpha$, and $p'_\beta=\sqrt{z-\lambda_{\beta,j}}\,\hat{p}'_\beta$, as well as to assume that the indices (α,i) and (β,j) are such that $l'_{\alpha,i}\neq 0$ and $l''_{\beta,j}\neq 0$. The kernels above turn out to be holomorphic functions of z on a set where the conditions

$$\Big[\tilde{k}_\alpha^{(\gamma)}(\sqrt{z-\lambda_{\alpha,i}}\,\hat{p}_\alpha,q)\Big]^2-\lambda_{\alpha,i}\ \neq\ 0\,,\quad \Big[\tilde{k}_\beta^{(\gamma)}(\sqrt{z-\lambda_{\beta,j}}\,\hat{p}'_\beta,q')\Big]^2-\lambda_{\beta,j}\ \neq\ 0$$

are satisfied for any $\gamma\neq\alpha$, $\delta\neq\beta$, $q,q'\in\mathbb{R}^3$ and $\hat{p}_\alpha,\hat{p}'_\beta\in S^2$. With respect to the variables $\hat{p}_\alpha,\hat{p}'_\beta$, the kernels (4.43) and (4.44) are real – analytic with these conditions. Besides, in the case of the potentials (2.3), the arguments of the functions $\tilde{\psi}_{\alpha,i}$ and $\psi_{\beta,j}$ have to belong to the strip W_{2b}, and the respective arguments of the T – matrices t_γ and t_δ to the strip W_b. It is easily to check that the mentioned conditions are satisfied for all the kernels (4.43) and (4.44) if

$$z\in\bigcap_{\substack{l'_{\nu,k}\neq 0,\\ l''_{\nu,k}\neq 0}}\Pi_b^{(\nu,k)}\,.$$

This completes the proof of Theorem 4.7. □

4.4. Analytic continuation of the matrices $\mathbf{J}_0M$ and $M\mathbf{J}_0^\dagger$, $\mathbf{J}_0M\mathbf{J}_0^\dagger$ and $\hat{\mathcal{T}}_{00}$, $\mathbf{J}_0M\Upsilon\Psi\mathrm{J}_1^\dagger$ and $\mathrm{J}_1\Psi^*\Upsilon M\mathbf{J}_0^\dagger$

Continuation of the half – on – shell matrices $(\mathbf{J}_0M)(z)$, $\big(M\mathbf{J}_0^\dagger\big)(z)$, $z=E\pm i0$, $E>0$, into a domain of complex z is considered in the sense of distributions over $\mathcal{O}\big(\mathbb{C}^6\big)$. For example, in the case of $M\mathbf{J}_0^\dagger$ we consider continuation of the bilinear form

$$\big(F,\ \big(M\mathbf{J}_0^\dagger\big)(E\pm i0)\big)\ \equiv\ \sum_{\alpha,\beta}\int_{\mathbb{R}^6}dP\int_{S^5}d\widehat{P}'\,F_\alpha(P)M_{\alpha\beta}\big(P,\pm\sqrt{E}\,\widehat{P}',E\pm i0\big)\,f_\beta\big(\widehat{P}'\big)$$

where $F=(F_1,F_2,F_3)$ with $F_\alpha\in\mathcal{O}\big(\mathbb{C}^6\big)$ and $f=(f_1,f_2,f_3)$ with $f_\alpha\in\widehat{\mathcal{H}}_0$.

When constructing continuation of this form and the form for $(\mathbf{J}_0M)(E\pm i0)$, we rely on Lemmas 4.5 and 4.6 describing domains of holomorphy of the function (4.25) in the case where the argument P' belongs to the three – body energy shell (2.15) and therefore $p'_\beta=\sqrt{z}\,\nu'\hat{p}'_\beta$ with $\nu'\in[0,\,1]$. Using the statements of these lemmas we introduce the following definition.

Let $\widetilde{\Pi}_b^{(0)\pm}$, $\widetilde{\Pi}_b^{(0)\pm} \subset \mathbb{C}^{\pm}$, be domains complementary in $\mathbb{C}^{\pm}$ to the totality of circles having radii $r = c_{\alpha\beta}^2 |\lambda_{\alpha,j}| / \bigl(1 - c_{\alpha\beta}^4\bigr)$ and centered at the points $z_c = \lambda_{\alpha,j} / \bigl(1 - c_{\alpha\beta}^4\bigr)$ where $\alpha, \beta = 1, 2, 3,\ \beta \neq \alpha$, and $j = 1, 2, \ldots, n_\alpha$.

In the case of the potentials (2.3) the domain $\widetilde{\Pi}_b^{(0)\pm}$ has to satisfy for both signs "+" and "−" the following extra conditions

$$\operatorname{Re} z \ > \ -\frac{|s_{\alpha\beta}|^2 b^2}{(1+|c_{\alpha\beta}|)^2} \ + \ \frac{(1+|c_{\alpha\beta}|)^2}{4\,|s_{\alpha\beta}|^2 b^2}\,(\operatorname{Im} z)^2 \tag{4.45}$$

for all $\alpha, \beta = 1, 2, 3,\ \beta \neq \alpha$.

Theorem 4.10. *The kernels of the matrices* $\bigl(M\mathbf{J}_0^\dagger\bigr)(z)$ *and* $(\mathbf{J}_0 M)(z)$, $\ z = E \pm i0$, $E > 0$, *admit analytic continuation in* z *on the domains* $\widetilde{\Pi}_b^{(0)+}$ *and* $\widetilde{\Pi}_b^{(0)-}$, $\widetilde{\Pi}_b^{(0)\pm} \subset \mathbb{C}^{\pm}$, *respectively. The continuation of the kernels of the matrices* $\bigl(\mathcal{Q}^{(n)}\mathbf{J}_0^\dagger\bigr)(z)$ *and* $\bigl(\mathbf{J}_0\mathcal{Q}^{(n)}\bigr)(z)$ $(n \leq 3)$ *included in the representation* (4.7) *for* $M(z)$ *has to be understood in the sense of distributions over* $\mathcal{O}\bigl(\mathbb{C}^6\bigr)$. *At the same time the kernels*

$$\begin{aligned} &\mathcal{F}_{\alpha\beta}\bigl(P, \sqrt{z}\,\widehat{P}', z\bigr)\,, && \mathcal{I}_{\alpha,j;\,\beta}\bigl(p_\alpha, \sqrt{z}\,\widehat{P}', z\bigr)\,,\\ &\mathcal{J}_{\alpha;\,\beta,k}\bigl(P, \sqrt{z}\,\sqrt{\nu'}\,\hat{p}'_\beta, z\bigr)\,, && \mathcal{K}_{\alpha,j;\,\beta,k}\bigl(p_\alpha, \sqrt{z}\,\sqrt{\nu'}\,\hat{p}'_\beta, z\bigr)\\ &\alpha, \beta \ = \ 1,\,2,\,3, \quad j \ = \ 1,\,2,\,\ldots,\,n_\alpha\,, && k \ = \ 1,\,2,\,\ldots,n_\beta\,, \end{aligned} \tag{4.46}$$

of the matrices $\bigl(\mathcal{Q}^{(n)}\mathbf{J}_0^\dagger\bigr)(z)$ $(n \geq 4)$ *and* $\bigl(\mathcal{W}\mathbf{J}_0^\dagger\bigr)(z)$ *as well as the kernels*

$$\begin{aligned} &\mathcal{F}_{\alpha\beta}\bigl(\sqrt{z}\,\widehat{P}, P', z\bigr)\,, && \mathcal{I}_{\alpha,j;\,\beta}\bigl(\sqrt{z}\,\sqrt{\nu}\,\hat{p}_\alpha, P', z\bigr)\,,\\ &\mathcal{J}_{\alpha;\,\beta,k}\bigl(\sqrt{z}\,\widehat{P}, p'_\beta, z\bigr)\,, && \mathcal{K}_{\alpha,j;\,\beta,k}\bigl(\sqrt{z}\,\sqrt{\nu}\,\hat{p}_\alpha, p'_\beta, z\bigr) \end{aligned} \tag{4.47}$$

of the matrices $\bigl(\mathbf{J}_0\mathcal{Q}^{(n)}\bigr)(z)$ $(n \geq 4)$ *and* $(\mathbf{J}_0\mathcal{W})(z)$ *can be continued on the domains* $\widetilde{\Pi}_b^{(0)\pm}$ *as usual holomorphic functions of the variable* z. *Being Hölder functions of the variables* $\widehat{P}' \in S^5$ *or* $\sqrt{\nu'}\,\hat{p}'_\beta$, $0 \leq \nu' \leq 1$, $\hat{p}'_\beta \in S^2$ $\bigl[\widehat{P} \in S^5$ *or* $\sqrt{\nu}\,\hat{p}_\alpha$, $0 \leq \nu \leq 1$, $\hat{p}_\alpha \in S^2\bigr]$ *with index* $\mu' \in (0,\ 1/8)$, *the kernels* (4.46) [*the kernels* (4.47)] *considered as functions of* $P \in \mathbb{R}^6$, $p_\alpha \in \mathbb{R}^3$ $\bigl(P' \in \mathbb{R}^6$, $p'_\beta \in \mathbb{R}^3\bigr)$, *can be embedded in their totality in* $\mathcal{B}_{\theta\mu}$ *with* θ *and* μ *being arbitrary numbers such that* $\theta \in (3/2, \theta_0)$ *and* $\mu \in (0, 1/8)$. *For* $|\operatorname{Im} z| \geq \delta > 0$ *one can take* $\mu = 1$.

Proof. Let us use the equations (2.9) for $m, n \geq 4$ keeping in mind that continuation of the kernels of $\bigl(\mathbf{J}_0\mathcal{Q}^{(m)}\bigr)(z)$ and $\bigl(\mathcal{Q}^{(n)}\mathbf{J}_0^\dagger\bigr)(z)$ for $m, n \leq 3$ is realized in the sense of distributions. At the same time, for $m, n \geq 4$ one can attach to the products $\bigl(\mathbf{J}_0\mathcal{Q}^{(m)}\bigr)(z)$ and $\bigl(\mathcal{Q}^{(n)}\mathbf{J}_0^\dagger\bigr)(z)$ an operator sense, $\bigl(\mathbf{J}_0\mathcal{Q}^{(m)}\bigr)(z) : \mathcal{G}_0 \to \widehat{\mathcal{G}}_0$, $\bigl(\mathcal{Q}^{(n)}\mathbf{J}_0^\dagger\bigr)(z) : \widehat{\mathcal{G}}_0 \to \mathcal{G}_0$. Thus, as in the case of (4.41) one may use Eqs. (2.9) as an instrument to obtain representations for the half−on−shell kernels $(\mathbf{J}_0 M)(z)$, $\bigl(M\mathbf{J}_0^\dagger\bigr)(z)$ as well as for the on−shell kernels $\widehat{\mathcal{T}}_{00}(z)$, $\widehat{\mathcal{T}}_{01}(z)$ and $\widehat{\mathcal{T}}_{10}$ in terms of the Faddeev components $M_{\alpha\beta}(P, P', z)$ with real $P, P' \in \mathbb{R}^6$.

The exposition will be given for the case of the matrices $\bigl(M\mathbf{J}_0^\dagger\bigr)(z)$.

First, we find easily that continuation on $\widetilde{\Pi}_b^{(0)\pm}$ of the form

$$\big(F, \big(\mathcal{Q}^{(0)}\mathbf{J}_0^\dagger\big)(z)f\big) \;=\; \sum_\alpha \big(F_\alpha, \big(\mathbf{t}_\alpha\mathbf{J}_0^\dagger\big)(z)f_\alpha\big)$$

for the matrix $\big(\mathbf{t}_\alpha\mathbf{J}_0^\dagger\big)(z)$ is described by the equalities

$$\begin{aligned}&\big(F_\alpha, \big(\mathbf{t}_\alpha \mathrm{J}_0^\dagger\big)(z)f_\alpha\big)\\ \equiv\;& \int_{\mathbb{R}^3} dk_\alpha \int_{S^2} d\hat{k}'_\alpha \int_{S^2} d\hat{p}'_\alpha \int_0^{\pi/2} d\omega'_\alpha \sin^2\omega'_\alpha \cos^2\omega'_\alpha\\ &\times\; t_\alpha\big(k_\alpha, \sqrt{z}\cos\omega'_\alpha \hat{k}'_\alpha, z\cos^2\omega'_\alpha\big) F_\alpha\big(k_\alpha, \pm\sqrt{z}\sin\omega'_\alpha\hat{p}'_\alpha\big)\cdot f_\alpha\big(\omega'_\alpha, \hat{k}'_\alpha, \hat{p}'_\alpha\big)\,,\end{aligned} \tag{4.48}$$

where $\omega'_\alpha, \hat{k}'_\alpha, \hat{p}'_\alpha$ are hyperspherical coordinates [46] of the point $\widehat{P}' \in S^5$, $\omega'_\alpha \in [0, \pi/2]$, $\hat{k}'_\alpha$, $\hat{p}'_\alpha \in S^2$. Note that

$$\widehat{P}' \;=\; \{\cos\omega'_\alpha \hat{k}'_\alpha, \sin\omega'_\alpha \hat{p}'_\alpha\} \quad \text{while} \quad d\widehat{P}' \;=\; \sin^2\omega'_\alpha \cos^2\omega'_\alpha\, d\omega'_\alpha\, d\hat{k}'_\alpha\, d\hat{p}'_\alpha$$

is a measure on S^5. A holomorphy domain of the function $\big(F_\alpha, \big(\mathbf{t}_\alpha \mathrm{J}_0^\dagger\big)(z)f\big)$ can be found from the conditions that the poles of the T – matrix $t_\alpha(\,\cdot\,, \,\cdot\,, z\cos^2\omega'_\alpha)$ corresponding to the discrete spectrum of the Hamiltonian h_α do not manifest themselves. In other words, one has to require the equalities

$$z\cos^2\omega'_\alpha \;=\; \lambda_{\alpha,j}\,, \quad \alpha \;=\; 1,\,2,\,3, \quad j \;=\; 1, 2, \ldots, n_\alpha\,, \tag{4.49}$$

to take place for no $\omega'_\alpha \in [0, \pi/2]$. Evidently, the last requirement is equivalent to making a cut along the ray $(-\infty, \lambda_{\max}]$. The poles (4.49) generate in (4.48) some integrals of the Cauchy type analogous to (3.2). Thus, a continuation of the function (4.48) through the cut $(-\infty, \lambda_{\max}]$ may be described using the representations (3.3) while each point $\lambda_{\alpha,j}$, $j = 1, 2, \ldots, n_\alpha$ turns into a branch point of the Riemann surface of the function (4.48).

In the case of the potentials (2.3), additionally to the cut $(-\infty, \lambda_{\max}]$, there appear additional restrictions on the domain of holomorphy of this function following from a requirement for the second argument of the T – matrix t_α to belong to the strip W_b, $|\operatorname{Im}\sqrt{z}\cos\omega'_\alpha \hat{k}'_\alpha| < b$ for all $\omega'_\alpha \in [0, \pi/2]$, $\hat{k}'_\alpha \in S^2$. This means that z has to be such that $|\operatorname{Im}\sqrt{z}| < b$, i. e., $z \in \mathcal{P}_b$ [see Eq. (3.6)].

Note that for $z \neq \lambda \pm i0$, $\lambda \leq \lambda_{\max}$ one can substitute any element $f \in \widehat{\mathcal{H}}_0$ in the form (4.48).

We shall consider formulas (4.48) with $\alpha = 1, 2, 3$ as a definition of analytic continuation of the matrix $\big(\mathbf{t}\mathbf{J}_0^\dagger\big)(z)$ on a domain of complex z. Up to now, we have established that immediate continuation of $\big(\mathbf{t}\mathbf{J}_0^\dagger\big)(z)$ described by the formulas (4.48) is possible on the domain $\mathcal{P}_b \setminus (-\infty, \lambda_{\max}]$.

Further, using Lemma 4.6 we find that the analytic continuation of the form $\big(F, \big(\mathcal{Q}^{(1)}\mathbf{J}_0^\dagger\big)(E \pm i0)f\big)$ on a domain of complex $z \in \mathbb{C}^\pm$ is given by

$$\big(F, \big(\mathcal{Q}^{(1)}\mathbf{J}_0^\dagger\big)(z)f\big) \;=\; \sum_{\alpha,\beta,\,\alpha\neq\beta} Q^\pm_{1,\alpha\beta}(z) \,+\, Q^\pm_{2,\alpha\beta}(z) \tag{4.50}$$

where

$$
\begin{aligned}
Q^{\pm}_{1,\alpha\beta}(z) = \pm \frac{\sqrt{z}}{4}\frac{1}{|s_{\alpha\beta}|} \int\limits_{\mathbb{R}^3} dk_\alpha \int\limits_{S^2} d\hat{p}_\alpha \int\limits_{S^2} dk'_\beta \int\limits_{S^2} d\hat{p}'_\beta \int\limits_0^1 d\nu\,\sqrt{\nu} \\
\times \int\limits_0^1 d\nu'\,\sqrt{\nu'}\,\sqrt{1-\nu'}\,\frac{F_\alpha\big(k_\alpha, \sqrt{z}\,\sqrt{\nu}\,\hat{p}_\alpha\big)\cdot f_\beta\big(\sqrt{1-\nu'}\,\hat{k}'_\beta, \sqrt{\nu'}\,\hat{p}'_\beta\big)}{\nu+\nu'-2c_{\alpha\beta}\sqrt{\nu}\,\sqrt{\nu'}\,\big(\hat{p}_\alpha,\hat{p}'_\beta\big)-s^2_{\alpha\beta}\mp i0} \\
\times\; t_\alpha\big(k_\alpha, \tilde{k}^{(\beta)}_\alpha\big(\sqrt{z}\,\sqrt{\nu}\hat{p}_\alpha, \sqrt{z}\,\sqrt{\nu'}\,\hat{p}'_\beta\big), z(1-\nu)\big) \\
\times\; t_\beta\big(\tilde{k}^{(\alpha)}_\beta(\sqrt{z}\,\sqrt{\nu'}\,\hat{p}'_\beta, \sqrt{z}\,\sqrt{\nu}\,\hat{p}_\alpha\big), \sqrt{z}\,\sqrt{1-\nu'}\,\hat{k}'_\beta, z(1-\nu')\big)
\end{aligned}
\tag{4.51}
$$

and

$$
\begin{aligned}
& Q^{\pm}_{2,\alpha\beta}(z) \\
= \; & \pm\frac{1}{4}\cdot\frac{1}{|s_{\alpha\beta}|} \int\limits_{\mathbb{R}^3} dk_\alpha \int\limits_{S^2} d\hat{p}_\alpha \int\limits_{S^2} dk'_\beta \int\limits_{S^2} d\hat{p}'_\beta \int\limits_{\Gamma^{\pm}_z} d\rho\,\sqrt{\rho} \\
& \times \int\limits_0^1 d\nu'\,\sqrt{\nu'}\,\sqrt{1-\nu'}s\,\frac{F_\alpha(k_\alpha, \pm\sqrt{\rho}\,\hat{p}_\alpha)\cdot f_\beta\big(\sqrt{1-\nu'}\,\hat{k}'_\beta, \sqrt{\nu'}\,\hat{p}'_\beta\big)}{\rho+z\nu'-2c_{\alpha\beta}\sqrt{z}\,\sqrt{\rho}\,\sqrt{\nu'}\,\big(\hat{p}_\alpha,\hat{p}'_\beta\big)-s^2_{\alpha\beta}z} \\
& \times\; t_\alpha\big(k_\alpha, \tilde{k}^{(\beta)}_\alpha\big(\pm\sqrt{\rho}\,\hat{p}_\alpha, \sqrt{z}\,\sqrt{\nu'}\hat{p}'{}_\beta\big), z-\rho\big) \\
& \times\; t_\beta\big(\tilde{k}^{(\alpha)}_\beta\big(\sqrt{z}\,\sqrt{\nu'}\,\hat{p}'_\beta, \pm\sqrt{\rho}\,\hat{p}_\alpha\big), \sqrt{z}\,\sqrt{1-\nu'}\,\hat{k}'_\beta, z(1-\nu')\big)\,.
\end{aligned}
\tag{4.52}
$$

Here, by Γ^+_z (Γ^-_z) we understand the path of integration beginning at the point z and going clockwise (counterclockwise) along the circumference $C_{|z|}$ having radius $|z|$ and centered in the origin. After the path crosses the real axis, it goes further along this axis so that the rest of Γ^+_z (Γ^-_z) consists of the points $\rho=\lambda+i0$ $(\rho=\lambda-i0)$, $\lambda\in(|z|,+\infty)$. It should be noted that as $\Gamma^{\pm}_z$ one can also choose arbitrary equivalent paths having no intersections with the line segment $[0,\, z]$ except at the point z and with the circle of radius $c^2_{\alpha\beta}|z|$ centered at the origin which are spoken about in Lemma 4.6.

One can find easily that if $f\big(\widehat{P}\big)$ is a Hölder function with the smoothness index $\mu>0$ then the functions $Q^+_{1,\alpha\beta}(z)$ and $Q^+_{2,\alpha\beta}(z)$ $\big[Q^-_{1,\alpha\beta}(z)$ and $Q^-_{2,\alpha\beta}(z)\big]$ are holomorphic in a domain of the upper half−plane $\mathbb{C}^+$ [of the lower half−plane $\mathbb{C}^-$], a boundary of which is determined by the numerators of the integrands in (4.51) and (4.52). In the case of the functions $Q^{\pm}_{1,\alpha\beta}(z)$ this fact does not require special explanation.

Concerning the functions $Q^{\pm}_{2,\alpha\beta}(z)$ we find that, according to Lemma 4.6, the denominators of the respective expressions under the integration signs may become zero for $\rho=z$ only. The latter is possible for the points corresponding to $\nu'=c^2_{\alpha\beta}$ and $\eta=(\hat{p}_\alpha,\hat{p}'_\beta)=-1$. Consequently, it suffices to check holomorphy of the integral in a small vicinity of this point; namely, the integral

$$
B(z) \;=\; \int\limits_{c^2_{\alpha\beta}-\varepsilon}^{c^2_{\alpha\beta}+\varepsilon} d\nu' \int\limits_{-1}^{-1+\delta} d\eta \int\limits_{z}^{z+z\zeta} d\rho\,\frac{\tilde{f}(\nu',\eta,\rho,z)}{\rho+z\nu'-2c_{\alpha\beta}\sqrt{z}\,\sqrt{\rho}\,\sqrt{\nu'}\,\eta-s^2_{\alpha\beta}z}
$$

where $\tilde f$ stands for the numerator of $Q^{\pm}_{2,\alpha\beta}(z)$ and ζ is a certain complex number, $0<|\zeta|<1$, $\arg\zeta\sim-\dfrac{\pi}{2}$. At the same time ε,δ are sufficiently small positive numbers, $0<\varepsilon<\min\left\{c^2_{\alpha\beta},s^2_{\alpha\beta}\right\}$ and $0<\delta<1$.

Let us rewrite $\tilde f$ in the form $\tilde f(\nu',\eta,\rho,z)=\tilde f\big(c^2,-1,z,z\big)+\delta\tilde f(\nu',\eta,\rho,z)$. Since $\delta\tilde f(c^2,-1,z,z)=0$, a contribution of the summand $\delta\tilde f$ to B is a holomorphic function. In the term generated by the summand $\tilde f(c^2,-1,z,z)$, the latter may be transferred through the integration sign. Making the substitution $\rho=z\nu'$ in the remaining integral one gets

$$\widetilde B(z)\;=\;\frac{1}{z}\int\limits_{c^2_{\alpha\beta}-\varepsilon}^{c^2_{\alpha\beta}+\varepsilon}d\nu\int\limits_{-1}^{-1+\delta}d\eta\int\limits_{1}^{1+\zeta}d\nu'\,\frac{1}{\nu+\nu'-2c_{\alpha\beta}\sqrt{\nu}\sqrt{\nu'}\,\eta-s^2_{\alpha\beta}}$$

where the integral converges being independent of z at all.

Thus, we show the form $\Big(F_\alpha,\big(\mathcal{Q}^{(1)}\mathbf{J}_0^\dagger\big)_{\alpha\beta}(E\pm i0)f_\beta\Big)$ admits, in correspondence with the sign "±", analytic continuation in z both in $\mathbb{C}^+$ and $\mathbb{C}^-$ on the domains of holomorphy of the above nominators.

Boundaries of these domains are found from those requirements that the poles of the T – matrices $t_\alpha(\,\cdot\,,\,\cdot\,,z(1-\nu))$ and $t_\beta(\,\cdot\,,\,\cdot\,,z(1-\nu'))$ which are present in the integral (4.51) do not appear in the above domains. Also, we require the same for the poles of the T – matrices $t_\alpha(\,\cdot\,,\,\cdot\,,z-\rho)$ which appear in the integral (4.52). If $z\notin(-\infty,\lambda_{\max}]$ then the conditions $z(1-\nu)=\lambda_{\alpha,j},\ j=1,2,\ldots,n_\alpha,\ z(1-\nu')=$ $=\lambda_{\beta,k},\ k=1,2,\ldots,n_\beta$ of appearance of the poles of the T – matrices $t_\alpha(\,\cdot\,,\,\cdot\,,z(1-\nu))$ and $t_\beta(\,\cdot\,,\,\cdot\,,z(1-\nu'))$ are valid for no $\nu,\nu'\in[0,\,1]$. The appearance conditions $z-\rho=\lambda_{\alpha,j}$, $j=1,2,\ldots,n_\alpha$, of the poles of $t_\alpha(\,\cdot\,,\,\cdot\,,z-\rho)$ may be realized only if the paths $\Gamma^\pm_z$ include more than one fourth of the circumference $C_{|z|}$. However, their contribution to $Q^\pm_{2,\alpha\beta}(z)$ arising when the points $\rho=z-\lambda_{\alpha,j}$ cross the contours $\Gamma^\pm_z$ may always be taken into account using the residue theorem. We shall not present here respective formulas. Note only that taking residues at the points $\rho=z-\lambda_{\alpha,j}$ transforms the minor three – body pole singularities of the integrand of $Q^\pm_{2,\alpha\beta}(z)$ into those of type $\big(z-\lambda_{\alpha,j}+z\nu'-2c_{\alpha\beta}\sqrt{z}\,\sqrt{z-\lambda_{\alpha,j}}\,\sqrt{\nu'}\,\eta-s^2_{\alpha\beta}z\big)^{-1}$. The location of such singularities is described by Lemma 4.5. As a result one finds that there are sets $\widetilde\Pi^{(0)\pm}_b$ which are holomorphy domains of the form (4.50). It should be noted only that the extra conditions (4.45) arise as a result of the requirements

$$\big|\operatorname{Im}\tilde k^{(\beta)}_\alpha\big(\sqrt{z}\,\sqrt{\nu}\,\hat p_\alpha,\sqrt{z}\,\nu'\,\hat p'_\beta\big)\big|\;<\;b\,,\qquad\big|\operatorname{Im}\tilde k^{(\alpha)}_\beta\big(\sqrt{z}\,\sqrt{\nu'}\,\hat p'_\beta,\sqrt{z}\,\nu\,\hat p_\alpha\big)\big|\;<\;b\,,$$

$$\big|\operatorname{Im}\tilde k^{(\beta)}_\alpha\big(\pm\sqrt{\rho}\,\hat p_\alpha,\sqrt{z}\,\nu'\,\hat p'_\beta\big)\big|\;<\;b\,,\qquad\big|\operatorname{Im}\tilde k^{(\alpha)}_\beta\big(\sqrt{z}\,\sqrt{\nu'}\,\hat p'_\beta,\pm\sqrt{\rho}\,\hat p_\alpha\big)\big|\;<\;b\,,$$

where $\nu,\nu'\in[0,\,1]$, $\hat p_\alpha,\hat p'_\beta\in S^2$, $\rho\in\Gamma^\pm_z$, and the condition

$$\big|\operatorname{Im}\big(\sqrt{z}\,\sqrt{1-\nu'}\,\hat k'_\beta\big)\big|\;<\;b\,,\quad\hat k'_\beta\in S^2\,,$$

which have to be satisfied for the arguments of the T – matrices t_α and t_β appearing in the expressions of $Q^\pm_{1,\alpha\beta}$ and $Q^\pm_{2,\alpha\beta}$ under the integration signs (since these arguments have to belong to the analyticity strip W_b).

The matrix $\big(\mathcal{Q}^{(2)}\mathbf{J}_0^\dagger\big)(z)$, $z = E \pm i0,\ E > 0$, corresponding to the second iteration of the absolute term of (2.7), has the kernels

$$\big(\mathcal{Q}^{(2)}\mathbf{J}_0^\dagger\big)\big(P, \widehat{P}', z\big) \;=\; \sum_{\gamma\neq\alpha,\,\gamma\neq\beta} Q^{(2)}_{\alpha\gamma\beta,0}\big(P, \widehat{P}', z\big)$$

where

$$\begin{aligned} &Q^{(2)}_{\alpha\gamma\beta,0}\big(P, \widehat{P}', z\big) \\ &= \frac{1}{|s_{\alpha\gamma}||s_{\gamma\beta}|}\int_{\mathbb{R}^3} dq\, t_\alpha\big(k_\alpha, \tilde{k}_\alpha^{(\gamma)}(p_\alpha, q), z - p_\alpha^2\big) \\ &\times\ \frac{t_\gamma\big(\tilde{k}_\gamma^{(\alpha)}(q, p_\alpha), \tilde{k}_\gamma^{(\beta)}(q, p'_\beta), z - q^2\big)}{\big(p_\alpha^2 + q^2 - 2c_{\alpha\gamma}(p_\alpha, q) - s_{\alpha\gamma}^2 z\big)\,\big({p'_\beta}^2 + q^2 - 2c_{\beta\gamma}(p'_\beta, q) - s_{\beta\gamma}^2 z\big)} \\ &\times\ t_\beta\big(\tilde{k}_\beta^{(\gamma)}(p'_\beta, q), k'_\beta, z - {p'_\beta}^2\big) \end{aligned} \tag{4.53}$$

with $\gamma \neq \alpha$, $\gamma \neq \beta$ and $P' = \pm\sqrt{E}\widehat{P}'$. The existence of analytic continuation of these kernels onto domains $\widetilde{\Pi}_b^{(0)\pm}$ may be proved, in the sense of distributions over $\mathcal{O}\big(\mathbb{C}^6\big)$, following the same scheme as for the kernels of the matrix $\big(\mathcal{Q}^{(1)}\mathbf{J}_0^\dagger\big)(z)$. However, it follows from the results of [42], [46] that these kernels have "better" properties than those of $\big(\mathcal{Q}^{(1)}\mathbf{J}_0^\dagger\big)(z)$. We have in mind now the fact that the components $\mathcal{F}_{\alpha\beta}(P, P', z)$, $\mathcal{I}_{\alpha,j;\,\beta}(p_\alpha, P', z)$, $\mathcal{J}_{\alpha;\,\beta,k}\big(P, p'_\beta, z\big)$ and $\mathcal{K}_{\alpha,j;\,\beta,k}\big(p_\alpha, p'_\beta, z\big)$, $P, P' \in \mathbb{R}^6$, $p_\alpha, p'_\beta \in \mathbb{R}^3$, of the iteration $\mathcal{Q}^{(2)}(z)$ of the absolute term of Eq. (2.7) turn out to be functions having weaker singularities than the components of $\mathcal{Q}^{(1)}(z)$. In particular, the main singularities of the kernel $\mathcal{F}_{\alpha\beta}(P, P', z)$ are described by

$$\frac{\pi^2 i}{D}\Big\{\mathsf{f}(\mathrm{a})\ln\big(\sqrt{\xi} + \sqrt{\zeta} + D\big) - \mathsf{f}(\mathrm{b})\ln\big(\sqrt{\xi} + \sqrt{\zeta} - D\big)\Big\} \tag{4.54}$$

with $\mathrm{a} = c_{\alpha\gamma}p_\alpha$, $\mathrm{b} = c_{\beta\gamma}p'_\beta$, $\xi = s_{\alpha\gamma}^2(z - p_\alpha^2)$, $\zeta = s_{\beta\gamma}^2(z - {p'_\beta}^2)$ and $D = \sqrt{(\mathrm{a}-\mathrm{b})^2}$ for $\gamma \neq \alpha$, $\gamma \neq \beta$. The notation $\mathsf{f}(q)$ is used here for the numerator of the expression under the integration sign in (4.54) after the replacements $t_\alpha \to \tilde{t}_\alpha$ and $t_\beta \to \tilde{t}_\beta$. The expressions for the main singularities of the kernels $\mathcal{I}_{\alpha,j;\,\beta}(p_\alpha, P', z)$, $\mathcal{J}_{\alpha;\,\beta,k}(P, p'_\beta, z)$, and $\mathcal{K}_{\alpha,j;\,\beta,k}(p_\alpha, p'_\beta, z)$ may also be represented in the form (4.54) but one has to take $\mathsf{f}(q)$ as the numerator of the expression in (4.54) replacing $t_\alpha \to \tilde{\phi}_\alpha\big(\tilde{k}_\alpha^{(\gamma)}(p_\alpha, q)\big)$ and/or $t_\beta \to \phi_\beta\big(\tilde{k}_\beta^{(\gamma)}(p'_\beta, q)\big)$. It should be emphasized that the singularities (4.54) only appear for $p_\alpha^2 \le |z|$ simultaneously with ${p'_\beta}^2 \le |z|$.

Thereby, when continuing the form $\big(F, \big(\mathcal{Q}^{(2)}\mathbf{J}_0^\dagger\big)(z)f\big)$ we get for it the representations which differ from (4.50) – (4.52) mainly in the replacement of the distributions $\big\{z\big(\nu + \nu' - 2c_{\alpha\beta}\sqrt{\nu}\sqrt{\nu'}\,(\hat{p}_\alpha, \hat{p}'_\beta) - s_{\alpha\beta}^2 \mp i0\big)\big\}^{-1}$, $0 \le \nu \le 1$, $0 \le \nu' \le 1$, with functions

singular like

$$\begin{aligned}\frac{1}{z\left|c_{\alpha\gamma}\nu\hat{p}_\alpha-c_{\beta\gamma}\nu'\hat{p}'_\beta\right|}\\ \times\ \ln\frac{\sqrt{s^2_{\alpha\gamma}\big(1-\nu^2\big)}+\sqrt{s^2_{\beta\gamma}\big(1-\nu'^2\big)}+\left|c_{\alpha\gamma}\nu\hat{p}_\alpha-c_{\beta\gamma}\nu'\hat{p}'_\beta\right|}{\sqrt{s^2_{\alpha\gamma}\big(1-\nu^2\big)}+\sqrt{s^2_{\beta\gamma}\big(1-\nu'^2\big)}-\left|c_{\alpha\gamma}\nu\hat{p}_\alpha-c_{\beta\gamma}\nu'\hat{p}'_\beta\right|}.\end{aligned}\tag{4.55}$$

The kernels $\mathcal{F}_{\alpha\beta}(P,P',z)$, $\mathcal{I}_{\alpha,j;\,\beta}(p_\alpha,P',z)$, $\mathcal{J}_{\alpha;\,\beta,k}\big(P,p'_\beta,z\big)$, and $\mathcal{K}_{\alpha,j;\,\beta,k}\big(p_\alpha,p'_\beta,z\big)$ of the iteration $\mathcal{Q}^{(3)}(z)=\left(-\mathbf{t}(z)\mathbf{R}_0(z)\Upsilon\right)^3\mathbf{t}(z)$ are still singular. Though their singularities are weak we understand the continuation of the kernels $\big(\mathcal{Q}^{(3)}\mathbf{J}_0^\dagger\big)(z)$ on the domains $\widetilde{\Pi}_b^{(0)\pm}$ as before in the sense of distributions over $\mathcal{O}\big(\mathbb{C}^6\big)$. So, we realize this continuation following the same scheme as for the continuation of $\big(\mathcal{Q}^{(1)}\mathbf{J}_0^\dagger\big)(z)$ and $\big(\mathcal{Q}^{(2)}\mathbf{J}_0^\dagger\big)(z)$.

As mentioned above, the components $\mathcal{F}_{\alpha\beta}(P,P',z)$, $\mathcal{I}_{\alpha,j;\,\beta}(p_\alpha,P',z)$, $\mathcal{J}_{\alpha;\,\beta,k}(P,p'_\beta,z)$, and $\mathcal{K}_{\alpha,j;\,\beta,k}(p_\alpha,p'_\beta,z)$ of the subsequent iterations $\mathcal{Q}^{(n)}(z)$ turn out to have no singularities.

Using the representations for the three – body singularities of $\mathcal{Q}^{(2)}(z)$, one can show that the kernels $\mathcal{F}_{\alpha\beta}\big(P,\sqrt{z}\,\widehat{P}',z\big)$, $\mathcal{I}_{\alpha,j;\,\beta}\big(p_\alpha,\sqrt{z}\,\widehat{P}',z\big)$, $\mathcal{J}_{\alpha;\,\beta,k}\big(P,\sqrt{z}\,\sqrt{\nu'}\,\hat{p}'_\beta,z\big)$ and $\mathcal{K}_{\alpha,j;\,\beta,k}\big(p_\alpha,\sqrt{z}\,\sqrt{\nu'}\,\hat{p}'_\beta,z\big)$ being components of the matrix $\big(\mathcal{Q}^{(4)}\mathbf{J}_0^\dagger\big)(z)$, turn out at $P\in\mathbb{R}^6$, $p_\alpha\in\mathbb{R}^3$, $\hat{P}'\in S^5$, $\hat{p}'_\beta\in S^2$ and $0\le\nu'\le 1$ to be holomorphic functions of $z\in\widetilde{\Pi}_b^{(0)\pm}$. The aggregate of these kernels may be embedded for fixed $\widehat{P}'$, $\hat{p}'_\beta$ and ν' into the Banach space $\mathcal{B}_{\theta\mu}$, $\theta<\theta_0$, $\mu<1/8$. And this aggregate becomes a function continuous in z with respect to the norm in $\mathcal{B}_{\theta\mu}$ right up to the edges of the cut $z=E\pm i0$, $E>0$. All the subsequent iterations $\big(\mathcal{Q}^{(n)}\mathbf{J}_0^\dagger\big)(z)$, $n\ge 5$, possess this property, too.

Assertions similar to those obtained concerning the continuation of the matrices $\big(\mathcal{Q}^{(n)}\mathbf{J}_0^\dagger\big)(z)$ also hold as well for the matrices $\big(\mathbf{J}_0\mathcal{Q}^{(n)}\big)(z)$.

Now, we use the equations (2.9) for $m,n\ge 4$ and thereby complete the proof. □

In the same way as Theorems 4.7 – 4.10, the two following assertions may be proved.

Theorem 4.11. *The matrix $\big(\mathbf{J}_0M\mathbf{J}_0^\dagger\big)(z)$ (the operator $\big(\mathrm{J}_0T\mathrm{J}_0^\dagger\big)(z)$) admits analytic continuation in z from the edges of the cut $z=E\pm i0$, $E>0$, to the domains $\widetilde{\Pi}_b^{(0)\pm}\in\mathbb{C}^\pm$ as a bounded operator in $\widehat{\mathcal{G}}_0$ (in $\widehat{\mathcal{H}}_0$). In addition, $\big(\mathbf{J}_0M\mathbf{J}_0^\dagger\big)(z)$, $z\in\widetilde{\Pi}_b^{(0)\pm}$ admits the representation* [cf. (4.7)]

$$\big(\mathbf{J}_0M\mathbf{J}_0^\dagger\big)(z)\;=\;\sum_{n=0}^{3}\big(\mathbf{J}_0\mathcal{Q}^{(n)}\mathbf{J}_0^\dagger\big)(z)+\big(\mathbf{J}_0\mathcal{W}\mathbf{J}_0^\dagger\big)(z).$$

The operators $\big(\mathbf{J}_0\mathcal{Q}^{(0)}\mathbf{J}_0^\dagger\big)(z)$ and $\big(\mathbf{J}_0\mathcal{Q}^{(1)}\mathbf{J}_0^\dagger\big)(z)$ are bounded matrix operators in $\widehat{\mathcal{G}}_0$ with singular kernels. Having weakly singular kernels, the matrices $\big(\mathbf{J}_0\mathcal{Q}^{(n)}\mathbf{J}_0^\dagger\big)(z)$, $n=2,3$, are compact operators in $\widehat{\mathcal{G}}_0$. The kernels of the matrix $\big(\mathbf{J}_0\mathcal{W}\mathbf{J}_0^\dagger\big)(z)$ are Hölder functions of their arguments with the smoothness index $\mu\in\big(0,1/8\big)$.

Theorem 4.12. *The operators*

$$\big(\mathbf{J}_0 M\Upsilon\Psi \mathrm{J}_1^\dagger\big)(z) \;:\; \widehat{\mathcal{H}}_1 \;\longrightarrow\; \widehat{\mathcal{G}}_0\,, \quad \big(\mathrm{J}_1\Psi^*\Upsilon M\mathbf{J}_0^\dagger\big)(z) \;:\; \widehat{\mathcal{G}}_0 \;\longrightarrow\; \widehat{\mathcal{H}}_1\,,$$
$$\widehat{\mathcal{T}}_{01}(z) \;:\; \widehat{\mathcal{H}}_1 \;\longrightarrow\; \widehat{\mathcal{H}}_0\,, \qquad \widehat{\mathcal{T}}_{10}(z) \;:\; \widehat{\mathcal{H}}_0 \;\longrightarrow\; \widehat{\mathcal{H}}_1$$

admit analytic continuation from the edges of the cut $z = E \pm i0$, $E > 0$, onto the domains $\Pi_b^{(0)\pm} \subset \mathbb{C}^{\pm}$ including the points $z \in \widetilde{\Pi}_b^{(0)\pm}\bigcap_{\beta,j}\Pi_b^{(\beta,j)}$ satisfying the additional conditions

$$\mathrm{Re}\, z \;>\; \frac{|s_{\beta\gamma}|^2}{(1+|c_{\beta\gamma}|)^2}\,\lambda_{\beta,j} \;+\; \frac{(1+|c_{\beta\gamma}|)^2}{4\,|s_{\beta\gamma}|^2|\lambda_{\beta,j}|}\,(\mathrm{Im}\, z)^2$$

for any $\beta,\gamma = 1,2,3$, $\beta\neq\gamma$, and $j = 1,2,\ldots,n_\beta$. For all $z\in\Pi_b^{(0)\pm}$ including the boundary points $z = E\pm i0$, $E>0$, these operators are compact.

As a comment to Theorem 4.11 we present explicit formulas for the kernels of the operators $\big(\mathbf{J}_0\mathcal{Q}^{(0)}\mathbf{J}_0^\dagger\big)(z)$ and $\big(\mathbf{J}_0\mathcal{Q}^{(1)}\mathbf{J}_0^\dagger\big)(z)$.

The kernels of the first operator have the form

$$\big(\mathbf{J}_0\mathcal{Q}^{(0)}\mathbf{J}_0^\dagger\big)_{\alpha\beta}\big(\widehat{P},\widehat{P}',z\big) \;=\; \delta_{\alpha\beta}\big(\mathrm{J}_0\mathbf{t}_\alpha\mathrm{J}_0^\dagger\big)\big(\widehat{P},\widehat{P}',z\big)\,,\quad \alpha,\,\beta \;=\; 1,\,2,\,3\,,$$

where

$$\begin{aligned}\big(\mathrm{J}_0\mathbf{t}_\alpha\mathrm{J}_0^\dagger\big)\big(\widehat{P},\widehat{P}',z\big) \;&=\; t_\alpha\big(\sqrt{z}\cos\omega_\alpha\hat{k}_\alpha, \sqrt{z}\cos{\omega'}_\alpha\hat{k}'_\alpha, z\cos^2\omega_\alpha\big)\\ &\times\; \delta\big(\sqrt{z}\sin\omega_\alpha\hat{p}_\alpha - \sqrt{z}\sin{\omega'}_\alpha\hat{p}'_\alpha\big)\,.\end{aligned}\tag{4.56}$$

Here, ω_α, $\hat{k}_\alpha$, $\hat{p}_\alpha$ and ω'_α, $\hat{k}'_\alpha$, $\hat{p}'_\alpha$ are coordinates of the points $\widehat{P} = \{k_\alpha, p_\alpha\}$ and $\widehat{P}' = \{k'_\alpha, p'_\alpha\}$ on the hypersphere S^5. We mean here that

$$\delta\big(\sqrt{z}\sin\omega\hat{p} - \sqrt{z}\sin\omega'\hat{p}'\big) \;=\; \mathrm{Sign}\,\mathrm{Im}\,z\cdot\frac{\delta(\hat{p},\hat{p}')\delta(\omega-\omega')}{\big(\sqrt{z}\big)^3\sin^2\omega\cos\omega}\tag{4.57}$$

where $\delta(\hat{p},\hat{p}')$ is the kernel of the identity operator in $L_2\big(S^2\big)$. The denominator $\big(\sqrt{z}\big)^3\sin^2\omega\cos\omega$ of the right−hand side of Eq. (4.57) represents the analytic continuation of the Jacobian corresponding to respective substitution of variables.

Therefore the operator $\big(\mathrm{J}_0\mathbf{t}_\alpha\mathrm{J}_0^\dagger\big)(z)$ acts at $\mathrm{Im}\,z\neq 0$ on $f\in\widehat{\mathcal{H}}_0$ as

$$\begin{aligned}\big(\big(\mathrm{J}_0\mathbf{t}_\alpha\mathrm{J}_0^\dagger\big)(z)f\big)\big(\widehat{P}\big) \;&=\; \frac{\mathrm{Sign}\,\mathrm{Im}\,z}{\big(\sqrt{z}\big)^3}\cdot\int\limits_{S^2} d\hat{k}'_\alpha\\ &\times\; t_\alpha\big(\sqrt{z}\cos\omega_\alpha\hat{k}_\alpha,\, \sqrt{z}\cos\omega_\alpha\hat{k}'_\alpha,\, z\cos^2\omega_\alpha\big)\\ &\times\; f\big(\cos\omega_\alpha\hat{k}'_\alpha, \sin\omega_\alpha\hat{p}_\alpha\big)\,.\end{aligned}\tag{4.58}$$

The operators $\big(\mathbf{J}_0\mathcal{Q}^{(1)}\mathbf{J}_0^\dagger\big)(z)$, $z\in\widetilde{\Pi}_b^{(0)\pm}$, have the kernels

$$\big(\mathbf{J}_0\mathcal{Q}^{(1)}\mathbf{J}_0^\dagger\big)_{\alpha\beta}\big(\widehat{P},\widehat{P}',z\big) = \frac{1}{z}\cdot\frac{1-\delta_{\alpha\beta}}{|s_{\alpha\beta}|}\cdot\frac{t_\alpha\big(k_\alpha, k_\alpha^{(\beta)}, z(1-\nu)\big)\;\, t_\beta\big(k_\beta^{(\alpha)}, k'_\beta, z(1-\nu')\big)}{\nu+\nu'-2c_{\alpha\beta}\sqrt{\nu}\sqrt{\nu'}\,(\hat{p}_\alpha,\hat{p}'_\beta) - s_{\alpha\beta}^2 \mp i0}\,,$$

where $k_\alpha = \sqrt{z}\,\sqrt{1-\nu}\,\hat{k}_\alpha$, $k'_\beta = \sqrt{z}\,\sqrt{1-\nu'}\,\hat{k}'_\beta$, $k_\alpha^{(\beta)} = \tilde{k}_\alpha^{(\beta)}\big(\sqrt{z}\,\sqrt{\nu}\,\hat{p}_\alpha,\ \sqrt{z}\,\sqrt{\nu'}\,\hat{p}'_\beta\big)$ and $k_\beta^{(\alpha)} = \tilde{k}_\beta^{(\alpha)}\big(\sqrt{z}\,\sqrt{\nu'}\,\hat{p}'_\beta,\ \sqrt{z}\,\sqrt{\nu}\,\hat{p}_\alpha\big)$. At the same time $\nu = \sin^2\omega_\alpha$ and $\nu' = \sin^2\omega'_\beta$.

The main singularities of the kernels $\big(\mathbf{J}_0\,\mathcal{Q}^{(2)}\mathbf{J}_0^\dagger\big)_{\alpha\beta}(\widehat{P},\widehat{P}',z)$ in $\widehat{P}$, $\widehat{P}'$ are described by Eqs. (4.55). The singularities of the kernels $\big(\mathbf{J}_0\,\mathcal{Q}^{(3)}\mathbf{J}_0^\dagger\big)_{\alpha\beta}(\widehat{P},\widehat{P}',z)$ are weaker.

Later, we shall use the notation

$$\Pi_{l^\pm}^{(\text{hol})} \equiv \Pi_b^{(0)\pm} \cap \Pi_{l^{(1)}}^{(\text{hol})}, \tag{4.59}$$

where $l^\pm = \big(l_0^\pm, l_{1,1}, \ldots, l_{1,n_1}, l_{2,1}, \ldots, l_{2,n_2}, l_{3,1}, \ldots, l_{3,n_3}\big)$ with $l_0^\pm = \pm 1$, $l_{\alpha,j} = 1$, $\alpha = 1,2,3$, $j = 1,2,\ldots,n_\alpha$, and $l^{(1)} = \big(0,\ l_{1,1}, \ldots, l_{1,n_1}, l_{2,1}, \ldots, l_{2,n_2}, l_{3,1}, \ldots, l_{3,n_3}\big)$ with the same $l_{\alpha,j}$ as in $l^\pm$. Remember that the sets $\Pi_{l^{(1)}}^{(\text{hol})} \equiv \Pi_{l^{(1)}l^{(1)}}^{(\text{hol})}$ were defined by Eqs. (4.34).

As follows from Theorems 4.7, 4.11 and 4.12, the total three – body scattering matrix $S(z)$, $z = E \pm i0$, $E > 0$, admits the analytic continuation as a holomorphic operator – valued function $S(z) : \widehat{\mathcal{H}}_0 \oplus \widehat{\mathcal{H}}_1 \to \widehat{\mathcal{H}}_0 \oplus \widehat{\mathcal{H}}_1$ on the domain $\Pi_{l^+}^{(\text{hol})} \subset \mathbb{C}^+$. For any $z \in \Pi_{l^+}^{(\text{hol})}$ the operator $S(z)$ is bounded. In equal degree the same is true for $S^\dagger(z)$.

5. Description of (part of) the three-body Riemann surface

By the three – body energy Riemann surface we mean the Riemann surface of the kernel $R(P,P',z)$ of the resolvent $R(z)$ of the Hamiltonian H consideration as a function of the parameter z, the energy of the three – body system.

One has to expect that this surface, like that of the free Green function $R_0(P,P',z)$, consists of an infinite number of sheets already because the threshold $z = 0$ is a logarithmic branching point. Actually the Riemann surface of $R(P,P',z)$ is much more complicated than that of $R_0(P,P',z)$, since besides $z = 0$ it has a lot of additional branching points. For example, the two – body thresholds $z = \lambda_{\alpha,j}$, $\alpha = 1,2,3$, $j = 1,2,\ldots,n_\alpha$ become square root branching points of this surface. Also, the resonances of pair subsystems turn into such points. Extra branching points are generated by boundaries of the supports of the function (4.25) singularities which were described in Lemmas 4.3, 4.4 and 4.5.

In the present paper we restrict ourselves to consider only of a "small" part of the total three – body Riemann surface for which we succeeded to find the explicit representations expressing analytic continuation of the Green function $R(P,P',z)$, the kernels of the matrix $M(z)$, as well as the scattering matrix $S(z)$ in terms of the physical sheet [see respective formulas (7.34), (8.1) and (9.1)]. Namely, in the Riemann surface of $R(P,P',z)$ we consider two neighboring "three – body" unphysical sheets immediately joint with the physical one along the three – body branch of the continuous spectrum $[0, +\infty)$. In addition, we examine all the "two – body" unphysical sheets, i. e., the sheets where the parameter z may be carried if going around the two – body thresholds $z = \lambda_{\alpha,j}$, $\alpha = 1,2,3$, $j = 1,2,\ldots,n_\alpha$, is permitted but crossing the ray $[0, +\infty)$ is forbidden. Evidently, the part of the three – body surface described includes all the sheets neighboring the physical one. The neighboring sheets are of most

interest in applications, since only resonances situated on these ones are accessible for immediate experimental observation.

We give a concrete description of the part under consideration using the auxiliary vector−function $\mathsf{f}(z) = (\mathsf{f}_0(z), \mathsf{f}_1(z), \mathsf{f}_2(z), \mathsf{f}_3(z))$, where $\mathsf{f}_0(z) = \ln z$ while

$$\mathsf{f}_\alpha(z) \;=\; \left((z-\lambda_{\alpha,1})^{1/2}, (z-\lambda_{\alpha,2})^{1/2}, \ldots, (z-\lambda_{\alpha,n_\alpha})^{1/2}\right), \quad \alpha \;=\; 1,\, 2,\, 3\,,$$

are again vector−functions.

The Riemann surface of $\mathsf{f}(z)$ consists of an infinite number of copies of the complex plane $\mathbb{C}'$ cut along the ray $[\lambda_{\min}, +\infty)$. These sheets are pasted together in a suitable way along edges of the cut segments between neighboring points in the set of thresholds $\lambda_{\alpha,j}$, $\alpha = 1,2,3,\ j = 1,2,\ldots,n_\alpha$ and $\lambda_0 = 0$. The sheets $\Pi_{l_0 l_1 l_2 l_3}$ are identified by indices of branches of the functions $\mathsf{f}_0(z) = \ln z$ and $\mathsf{f}_{\alpha,j}(z) = \left(z-\lambda_{\alpha,j}\right)^{1/2}$ in such a manner that l_0 is integer and l_α, $\alpha = 1,2,3$ are multi−indices, $l_\alpha = (l_{\alpha,1}, l_{\alpha,2}, \ldots, l_{\alpha,n_\alpha})$, $l_{\alpha,j} = 0,1$. For the main branch of the function $\mathsf{f}_{\alpha,j}(z)$, $\alpha = 1,2,3,\ j = 1,2,\ldots,n_\alpha$, we take $l_{\alpha,j} = 0$, and otherwise $l_{\alpha,j} = 1$. In case there exist coinciding thresholds, i.e., $\lambda_{\alpha,i} = \lambda_{\beta,j}$ at $\alpha \neq \beta$ and/or $i \neq j$ (this means that the discrete spectra of the pair Hamiltonians coincide at least partly for two pair subsystems or at least one of the pair subsystems has a multiple discrete spectrum), then for each sheet $\Pi_{l_0 l_1 l_2 l_3}$ the indices $l_{\alpha,i}$ and $l_{\beta,j}$ coincide too, $l_{\alpha,i}{=}l_{\beta,j}$. As l_0 we choose the number of the function $\ln z$ branch, $\ln z = \ln|z| + i\varphi_0 + i2\pi l_0$ with φ_0, the argument of z, $z = |z|\,\mathrm{e}^{i\varphi_0}$, $\varphi_0 \in [0, 2\pi)$. The sheets $\Pi_{l_0 l_1 l_2 l_3}$ are pasted together (along edges of the cut) in such a way that if the parameter z going from the sheet $\Pi_{l_0 l_1 l_2 l_3}$ crosses the interval between two neighboring thresholds $\lambda_{\alpha,i}$ and $\lambda_{\beta,j}$, $\lambda_{\alpha,i} < \lambda_{\beta,j}$ (or $\lambda_{\max}$ and λ_0) then it goes over the sheet $\Pi_{l'_0 l'_1 l'_2 l'_3}$ where the indices $l_{\gamma,k}$ corresponding to $\lambda_{\gamma,k} \leq \lambda_{\alpha,i}$ ($\lambda_{\gamma,k} \leq \lambda_{\max}$) changed by 1. If $l_{\gamma,k} = 0$, then $l'_{\gamma,k} = 1$; if $l_{\gamma,k} = 1$, then $l'_{\gamma,k} = 0$. The indices $l_{\gamma,k}$ for $\lambda_{\gamma,k} > \lambda_{\alpha,i}$ and l_0 stay unchanged: $l'_{\gamma,k} = l_{\gamma,k}$, $l'_0 = l_0$. In case the parameter z crosses the cut on the right from the three−body threshold λ_0 (at $E > \lambda_0$), then all the indices $l_{\gamma,k}$ change as it was described above. In addition, the index l_0 changes by 1, too. If at that, z crosses the cut from below, then $l'_0 = l_0 + 1$. Otherwise $l'_0 = l_0 - 1$. Further, by l we denote the multi−index $l = (l_0, l_1, l_2, l_3)$.

Thus, we have described the Riemann surface of the auxiliary vector−function $\mathsf{f}(z)$.

As mentioned above we shall consider only a part of the three−body Riemann surface which will be denoted by $\Re$. We include in $\Re$ all the sheets Π_l of the Riemann surface of the function $\mathsf{f}(z)$ with $l_0 = 0$. Also, we include in $\Re$ the upper half−plane $\operatorname{Im} z > 0$ of the sheet Π_l with $l_0 = +1$ and the lower half−plane $\operatorname{Im} z < 0$ of the sheet Π_l with $l_0 = -1$. For these parts we keep the previous notations Π_l, $l_0 = \pm 1$, assuming additionally that the cuts are made on them along the rays belonging to the set $Z_{\mathrm{res}} = \bigcup_{\alpha=1}^{3} Z_{\mathrm{res}}^{(\alpha)}$. Here, $Z_{\mathrm{res}}^{(\alpha)} = \left\{z : \, z = z_r\rho,\, 1 \leq \rho < +\infty,\, z_r \in \sigma_{\mathrm{res}}^{(\alpha)}\right\}$ is a totality of the rays beginning at the resonances $z_r \in \sigma_{\mathrm{res}}^{(\alpha)}$ of the pair subsystem α and going to infinity in the directions $\hat{z}_r = z_r/|z_r|$.

The sheet Π_l for which all the components of the multi−index l are zero, $l_0 = l_{\alpha,j} = 0$ $= 0$, $\alpha = 1,2,3,\ j = 1,2,\ldots,n_\alpha$, is called the physical sheet. The unphysical sheets Π_l with $l_0 = 0$ are called the two−body sheets since these ones may be reached by only going around two−body thresholds and it is not necessary to bypass the three−body threshold λ_0. The sheets Π_l at $l_0 = \pm 1$ are called the three−body sheets.

6. Analytic continuation of the Faddeev integral equations into unphysical sheets

A goal of the present section consists in a continuation into the unphysical sheets of the surface $\Re$ of the absolute terms and kernels of the Faddeev equations (2.7) and their iterations. Continuation is realized in the sense of generalized functions (distributions) over $\mathcal{O}(\mathbb{C}^6)$. Results of the continuation are represented in terms related to the physical sheet only.

By $L^{(\alpha)}$, $L^{(\alpha)} = L^{(\alpha)}(l)$, we denote the diagonal matrices formed of components $l_{\alpha,1}$, $l_{\alpha,2}$, ..., l_{α,n_α} of the multi – index l of the sheet $\Pi_l \subset \Re$:

$$L^{(\alpha)} = \operatorname{diag}\{l_{\alpha,1}, l_{\alpha,2}, \ldots, l_{\alpha,n_\alpha}\}.$$

Meanwhile $L_1(l) = \operatorname{diag}\left\{L^{(1)}, L^{(2)}, L^{(3)}\right\}$ and $L(l) = \operatorname{diag}\{L_0, L_1\}$ with $L_0 \equiv l_0$. Analogously,

$$\begin{aligned} A^{(\alpha)}(z) &= \operatorname{diag}\{A_{\alpha,1}(z), A_{\alpha,2}, \ldots, A_{\alpha,n_\alpha}(z)\}, \\ A_1(z) &= \operatorname{diag}\left\{A^{(1)}(z), A^{(2)}(z), A^{(3)}(z)\right\}. \end{aligned}$$

Thus $A(z) = \operatorname{diag}\{A_0(z), A_1(z)\}$.

By $\mathbf{s}_{\alpha,l}(z)$ we understand an operator defined in $\widehat{\mathcal{H}}_0$ by

$$\mathbf{s}_{\alpha,l}(z) = \hat{I}_0 + \mathrm{J}_0(z)\mathbf{t}_\alpha(z)\mathrm{J}_0^\dagger(z)A_0(z)L_0, \quad z \in \Pi_0. \tag{6.1}$$

It follows from Eq. (6.1) that $\mathbf{s}_{\alpha,l} = \hat{I}_0$ at $l_0 = 0$. If $l_0 = \pm 1$, then, according to Eqs. (4.56) – (4.58), the operator $\mathbf{s}_{\alpha,l}(z)$ is defined for $z \in \mathcal{P}_b \cap \mathbb{C}^\pm$ acting on $f \in \widehat{\mathcal{H}}_0$ by

$$(\mathbf{s}_{\alpha,l}(z)f)(\widehat{P}) = \int_{S^2} d\hat{k}'\, s_\alpha\big(\hat{k}_\alpha, \hat{k}'_\alpha, z\cos^2\omega\big) f\big(\cos\omega_\alpha \hat{k}'_\alpha, \sin\omega_\alpha \hat{p}_\alpha\big) \tag{6.2}$$

where ω_α, $\hat{k}_\alpha$, $\hat{p}_\alpha$ stand for the coordinates [46] of the point $\widehat{P}$ on the hypersphere S^5, $\omega_\alpha \in [0, \pi/2]$, $\hat{k}_\alpha$, $\hat{p}_\alpha \in S^2$ and $\widehat{P} = \{\cos\omega_\alpha \hat{k}_\alpha,\ \sin\omega_\alpha \hat{p}_\alpha\}$. By s_α we denote the scattering matrix (3.10) for the pair subsystem α. Here we have taken into account the fact that $l_0 \cdot \operatorname{Sign}\operatorname{Im} z = 1$ both for $l_0 = 1$ and $l_0 = -1$. Recall that at $l_0 = 1$ the set Π_l represents the upper half – plane and at $l_0 = -1$, the lower one (in accordance with our choice in Sec. 5 of the part $\Re$ of a total Riemann surface in the problem of three particles). Therefore, one can see that the operators $\mathbf{s}_{\alpha,l}$ are described by the same formula (6.2) in both three – body sheets Π_l, $l_0 = \pm 1$. As a matter of fact, $\mathbf{s}_{\alpha,l}$ represents the pair scattering matrix s_α rewritten in the three – body momentum space.

It follows immediately from Eq. (6.2) that if $z \in \mathcal{P}_b \cap \mathbb{C}^\pm \setminus Z_{\mathrm{res}}^{(\alpha)}$, then the bounded inverse operator $\mathbf{s}_{\alpha,l}^{-1}(z)$ exists and

$$(\mathbf{s}_{\alpha,l}^{-1}(z)f)(\widehat{P}) = \int_{S^2} d\hat{k}'\, s_\alpha^{-1}\big(\hat{k}_\alpha, \hat{k}'_\alpha, z\cos^2\omega_\alpha\big) f\big(\cos\omega_\alpha \hat{k}'_\alpha, \sin\omega_\alpha \hat{p}_\alpha\big)$$

with $s_\alpha^{-1}\big(\hat{k}, \hat{k}', \zeta\big)$ the kernel of the inverse scattering matrix $s_\alpha^{-1}(\zeta)$.

The operator $\mathbf{s}_{\alpha,l}^{-1}(z)$ becomes unbounded at the boundary points z situated on the edges of the cuts (the "resonance" rays) included in $Z_{\text{res}}^{(\alpha)}$.

Theorem 6.1. *The absolute terms* $\mathbf{t}_\alpha(P,P',z)$ *and kernels* $(\mathbf{t}_\alpha R_0)(P,P',z)$ *of the Faddeev equations* (2.7) *admit analytic continuation in the sense of distributions over* $\mathcal{O}(\mathbb{C}^6)$ *both into the two – body and three – body sheets* Π_l *of the Riemann surface* $\Re$. *The continuation into the sheet* Π_l, $l=(l_0,l_{1,1},\ldots,l_{1,n_1},l_{2,1},\ldots,l_{2,n_2},l_{3,1},\ldots,l_{3,n_3})$, $l_0 = = 0$, $l_{\beta,j}=0,1$, *or* $l_0=\pm1$, $l_{\beta,j}=1$ (*in the both cases* $\beta=1,2,3$, $j=1,2,\ldots,n_\beta$), *is written as*

$$\mathbf{t}_\alpha^l(z) \equiv \mathbf{t}_\alpha(z)\big|_{\Pi_l} = \mathbf{t}_\alpha - L_0A_0\mathbf{t}_\alpha \mathrm{J}_0^\dagger \mathbf{s}_{\alpha,l}^{-1}\mathrm{J}_0\mathbf{t}_\alpha - \Phi_\alpha \mathrm{J}^{(\alpha)t}L^{(\alpha)}A^{(\alpha)}\mathrm{J}^{(\alpha)}\Phi_\alpha^*, \tag{6.3}$$

$$[\mathbf{t}_\alpha(z)R_0(z)]\big|_{\Pi_l} = \mathbf{t}_\alpha^l(z)R_0^l(z), \tag{6.4}$$

where $R_0^l(z)\equiv R_0(z)\big|_{\Pi_l} = R_0(z)+L_0A_0(z)\mathrm{J}_0^\dagger(z)\mathrm{J}_0(z)$ *is the continuation* (3.6) *on* Π_l *of the free Green function* $R_0(z)$. *If* $l_0=0$ (*and hence* Π_l *is a two – body unphysical sheet*), *then the continuation in the form* (6.3), (6.4) *is possible on the whole sheet* Π_l. *For* $l_0=\pm1$, (*i. e., in the case where* Π_l *is a three – body sheet*) *the continuation in the form* (6.3), (6.4) *is possible on the domain* $\mathcal{P}_b\cap\Pi_l$. *All the kernels on the right – hand side of Eqs.* (6.3) *are taken in the physical sheet.*

Proof. We prove the theorem for the case of the most complicated continuation into the three – body unphysical sheets Π_l with $l_0=\pm1$. For the sake of definiteness we consider the case $l_0=+1$. For $l_0=-1$ the proof is quite analogous.

Let us consider at $z\in\Pi_0$, $\operatorname{Im} z<0$ the bilinear form

$$(f,\mathbf{t}_\alpha R_0(z)f') = \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dk' \int_{\mathbb{R}^3} dp\, \frac{t_\alpha(k,k',z-p^2)}{k'^2+p^2-z}\,\tilde f(k,k',p) \tag{6.5}$$

with $\tilde f(k,k',p)=f(k,p)f'(k',p)$, $f,f'\in\mathcal{O}(\mathbb{C}^6)$, $k=k_\alpha$, $k'=k'_\alpha$, $p=p_\alpha$. Making the substitutions $|k'|\to\rho=|k'|^2$, $|p|\to\lambda=z-|p|^2$ the integral (6.5) becomes

$$\frac14\int\limits_{\mathbb{R}^3} dk\int\limits_{S^2} d\hat k'\int\limits_{S^2} d\hat p\int\limits_{z-\infty}^{z} d\lambda\sqrt{z-\lambda}\int\limits_0^\infty d\rho\sqrt\rho\,\frac{t_\alpha(k,\sqrt\rho\hat k',\lambda)}{\rho-\lambda}\,\tilde f(k,\sqrt\rho\,\hat k',\sqrt{z-\lambda}\,\hat p). \tag{6.6}$$

The existence of an analytic continuation of the kernel $(\mathbf{t}_\alpha R_0)(z)$ into the sheet Π_l, $l_0=\pm1$ follows from the possibility of continuously deform the path of integration in the variable ρ to an arbitrary sector of the holomorphy domain $\mathcal{P}_b\cap\sigma_{\text{res}}^{(\alpha)}$ of the integrand in the variable λ in the way demonstrated in Fig. 1.

Besides, this is connected with the possibility when taking z from Π_0 to Π_l, $l_0=+1$ to make a necessary deformation of the integration path in λ in such a way that this path is separated from the integration contour in ρ.

To obtain the representation (6.4) at a concrete point $z=z_0$ we choose special final locations of the integration paths in the variables λ and ρ after their consistent deformation (see Fig. 2).

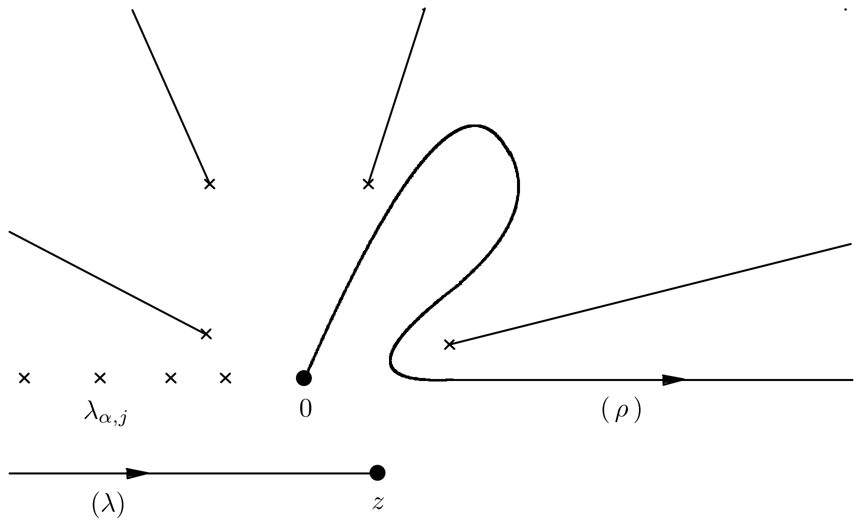

Figure 1: Deformation of the integration path in the variable ρ. The integration paths in ρ and λ are denoted by letters in brackets. The cross "×" denotes the eigenvalues $\lambda_{\alpha,j}$ of h_α on the negative half-axis of the physical sheet and the pair resonances belonging to the set $\sigma_{\rm res}^{(\alpha)}$ of the sheet Π_l, $l_0 = +1$. Also, the cuts on Π_l, $l_0 = +1$ beginning at the points of $\sigma_{\rm res}^{(\alpha)}$ are shown in the figure.

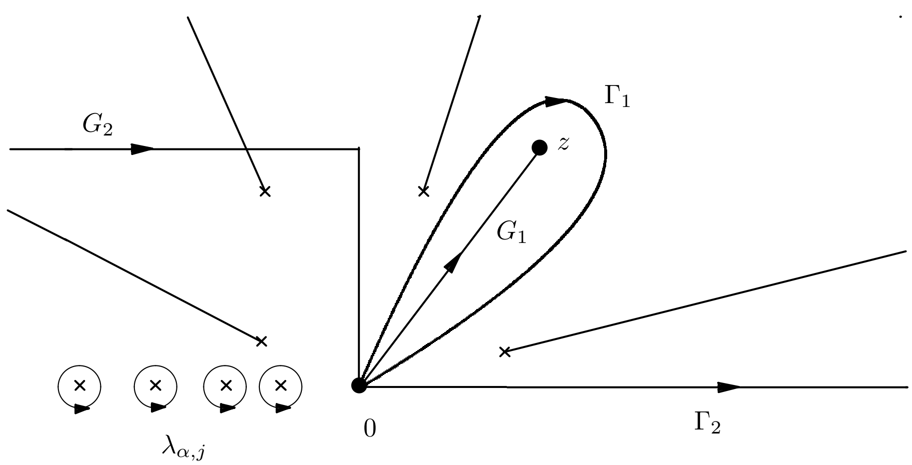

Figure 2: Final location of the integration paths in the variables ρ $(\Gamma_1 \cup \Gamma_2)$ and λ $(G_1 \cup G_2)$. The path Γ_1 represents a loop going clockwise around the path G_1, the line segment $[0,\ z]$; $\Gamma_2 = [0,\ +\infty)$; $G_2 = (z - \infty,\ i\,\mathrm{Im}\,z] \cup [i\,\mathrm{Im}\,z,\ 0)$.

The singularity of the inner integral (in the variable ρ) remains integrable after such a deformation due to the presence of the factor $\sqrt{\rho}$. As a whole, the integral (6.6) becomes

$$
\begin{aligned}
&\frac{1}{4}\int\limits_{\mathbb{R}^3} dk \int\limits_{S^2} d\hat{k}' \int\limits_{S^2} d\hat{p} \\
&\times \left\{ \int\limits_{G_1} d\lambda\sqrt{z-\lambda} \int\limits_{\Gamma_1\cup\Gamma_2} d\rho\sqrt{\rho}\,\frac{t'_\alpha\big(k,\sqrt{\rho}\,\hat{k}',\lambda\big)}{\rho-\lambda}\,\tilde{f}\big(k,\sqrt{\rho}\,\hat{k}',\sqrt{z-\lambda}\,\hat{p}\big) \right.\\
&+ \int\limits_{G_2} d\lambda\sqrt{z-\lambda} \int\limits_{\Gamma_1\cup\Gamma_2} d\rho\sqrt{\rho}\,\frac{t_\alpha\big(k,\sqrt{\rho}\,\hat{k}',\lambda\big)}{\rho-\lambda}\,\tilde{f}\big(k,\sqrt{\rho}\,\hat{k}',\sqrt{z-\lambda}\,\hat{p}\big) \\
&\left. + \sum_{j=1}^{n_\alpha} 2\pi i\,\sqrt{z-\lambda_{\alpha,j}} \int\limits_0^{+\infty} d\rho\sqrt{\rho}\,\frac{\phi_{\alpha,j}(k)\,\overline{\phi}_{\alpha,j}(k')}{\rho-\lambda_{\alpha,j}}\,\tilde{f}\big(k,\sqrt{\rho}\,\hat{k}',\sqrt{z-\lambda_{\alpha,j}}\,\hat{p}\big) \right\}
\end{aligned}
\tag{6.7}
$$

where t'_α denotes the pair T$-$matrix $t_\alpha(z)$ continued into the unphysical sheet (with respect to $t_\alpha(\lambda)$ the path G_1, $\lambda \in G_1$ just belongs to this sheet). The last term arises as a result of taking residues at the points $\lambda_{\alpha,j} \in \sigma_d(h_\alpha)$.

Evidently, a domain of the variable $z \in \Pi_l$, $l_0 = +1$, where one can continue the function (6.5) analytically in the form (6.7) is determined by the conditions $\Gamma_1 \subset \mathcal{P}_b$ and $\Gamma_1 \cap Z^{(\alpha)}_{\text{res}} = \emptyset$. These conditions can only be satified for $z \in \mathcal{P}_b$.

Note that the values of the inner integrals along Γ_1 are determined for $\lambda \in G_1$ by the residues at the points $\rho = \lambda$. At the same time $\int_{G_2} d\lambda \ldots \int_{\Gamma_1} \ldots = 0$, since at $\lambda \in G_2$ the functions under the integration sign are holomorphic in $\rho \in \operatorname{Int}\Gamma_1$. Therefore,

$$
\begin{aligned}
&\big(f, \mathbf{t}_\alpha R_0(z) f'\big)\big|_{z\in\Pi_l,\, l_0=+1} \\
=\;& \frac{1}{4}\int_{\mathbb{R}^3} dk \int_{S^2} d\hat{k}' \int_{S^2} d\hat{p} \\
&\times \left\{ \int\limits_{G_1} d\lambda\sqrt{z-\lambda}\,(-2\pi i)\sqrt{\lambda}\,t'_\alpha\big(k,\sqrt{\lambda}\,\hat{k}',\lambda\big)\tilde{f}\big(k,\sqrt{\lambda}\,\hat{k}',\sqrt{z-\lambda}\,\hat{p}\big) \right.\\
&+ \int\limits_{G_1} d\lambda\sqrt{z-\lambda} \int\limits_{\Gamma_2} d\rho\sqrt{\rho} \\
&\times \frac{t_\alpha\big(k,\sqrt{\rho}\,\hat{k}',\lambda\big) + \pi i\sqrt{\lambda}\,\tau_\alpha\big(k,\sqrt{\rho}\,\hat{k}',\lambda\big)}{\rho-\lambda}\,\tilde{f}\big(k,\sqrt{\rho}\,\hat{k}',\sqrt{z-\lambda}\,\hat{p}\big) \\
&+ \int\limits_{G_2} d\lambda\sqrt{z-\lambda} \int\limits_{\Gamma_2} d\rho\sqrt{\rho}\,\frac{t_\alpha\big(k,\sqrt{\rho}\,\hat{k}',\lambda\big)}{\rho-\lambda}\,\tilde{f}\big(k,\sqrt{\rho}\,\hat{k}',\sqrt{z-\lambda}\,\hat{p}\big) \\
&\left. + \sum_{j=1}^{n_\alpha} 2\pi i\sqrt{z-\lambda_{\alpha,j}} \int\limits_0^{+\infty} d\rho\sqrt{\rho}\,\frac{\phi_{\alpha,j}(k)\,\overline{\phi}_{\alpha,j}(k')}{\rho-\lambda_{\alpha,j}}\,\tilde{f}\big(k,\sqrt{\rho}\,\hat{k}',\sqrt{z-\lambda_{\alpha,j}}\,\hat{p}\big) \right\}.
\end{aligned}
\tag{6.8}
$$

In the second summand of Eq. (6.9) we have used the representation (3.11).

Joining the summands involving t_α on the physical sheet in a separate integral $\int_{G_1\cup G_2}\ldots$ and then using the holomorphy property of the integrand in λ we straighten the path $G_1\cup G_2$ turning it into the ray $(z-\infty, z]$. As a result we get the bilinear form corresponding to the product $\big(\mathbf{t}_\alpha R_0\big)(z)$ taken in the physical sheet.

The last term of the expression (6.9) corresponds to the kernel of the product $-\Phi_\alpha \mathrm{J}^{(\alpha)\dagger} L^{(\alpha)} A^{(\alpha)} \mathrm{J}^{(\alpha)} \Phi^* R_0$.

Returning the reminding summands involving t'_α and τ_α to the initial variables k', p' and utilizing then the definition (6.1), we find these summands correspond to the expression

$$L_0 A_0\Big[\mathbf{t}_\alpha - L_0 A_0 \mathrm{J}_0^\dagger \mathbf{s}_{\alpha,l}^{-1} \mathrm{J}_0 \mathbf{t}_\alpha\Big] \mathrm{J}_0^\dagger \mathrm{J}_0 \;-\; L_0 A_0 \mathbf{t}_\alpha \mathrm{J}_0^\dagger \mathbf{s}_{\alpha,l}^{-1} \mathrm{J}_0 \mathbf{t}_\alpha R_0\,.$$

Gathering the results obtained we reveal that the analytical continuation of $\mathbf{t}_\alpha R_0$ into the sheet Π_l, $l_0=+1$, reads

$$\begin{aligned}[\mathbf{t}_\alpha R_0(z)]\big|_{\Pi_l} \;&=\; \big(\mathbf{t}_\alpha - L_0 A_0 \mathbf{t}_\alpha \mathrm{J}_0^\dagger \mathbf{s}_{\alpha,l}^{-1} \mathrm{J}_0 \mathbf{t}_\alpha - \Phi_\alpha \mathrm{J}^{(\alpha)\dagger} L^{(\alpha)} A^{(\alpha)} \mathrm{J}^{(\alpha)} \Phi_\alpha^*\big) \\ &\quad\times\; \big(R_0 + L_0 A_0 \mathrm{J}_0^\dagger \mathrm{J}_0\big) + L_0 A_0 \Phi_\alpha \mathrm{J}^{(\alpha)\dagger} L^{(\alpha)} A^{(\alpha)} \mathrm{J}^{(\alpha)} \Phi_\alpha^* \mathrm{J}_0^\dagger \mathrm{J}_0\,.\end{aligned} \tag{6.9}$$

To be convinced of the factorization (6.4) it suffices to observe that the last summand of (6.10) is equal to zero. Indeed, for $\operatorname{Im} z \neq 0$ or $\operatorname{Im} z = 0$ and $z > \max_j \lambda_{\alpha,j}$ the following equalities hold

$$\big(\mathrm{J}^{(\alpha)} \Phi_\alpha^* \mathrm{J}_0^\dagger\big)(z) \;=\; 0\,, \quad \big(\mathrm{J}_0 \Phi_\alpha \mathrm{J}^{(\alpha)\dagger}\big)(z) \;=\; 0\,. \tag{6.10}$$

To prove, say, the first of them one can consider the j – th component of the matrix – column $\mathrm{J}^{(\alpha)}\Phi^* \mathrm{J}_0^\dagger$, $\big(\mathrm{J}^{(\alpha)}\Phi^* \mathrm{J}_0^\dagger\big)_j(z) \;=\; \mathrm{J}_{\alpha,j}(z)\langle\,\cdot\,, \phi_{\alpha,j}\rangle \mathrm{J}_0^\dagger(z)$. This component acts on $f \in L_2\big(S^5\big)$ as follows

$$\begin{aligned}&\big(\big(\mathrm{J}^{(\alpha)}\Phi^* \mathrm{J}_0^\dagger\big)_j f\big)\big(\hat{k}_\alpha, z\big) \\ &=\; \int dk''_\alpha\, \overline{\phi}_\alpha\big(k''_\alpha\big) \int d\widehat{P}' \delta\big(k''_\alpha - \sqrt{z}\cos\omega'_\alpha \hat{k}'_\alpha\big) \delta\big(\sqrt{z-\lambda_{\alpha,j}}\,\hat{p}_\alpha - \sqrt{z}\sin\omega'_\alpha \hat{p}'_\alpha\big) f\big(\widehat{P}'\big)\end{aligned}$$

where we use again the hyperspherical coordinates, $\widehat{P}' \sim \big(\omega'_\alpha, \hat{k}'_\alpha, \hat{p}'_\alpha\big)$. It is clear that only the points $\widehat{P}' \in S^5$ with $\sqrt{z}\sin\omega'_\alpha \hat{p}'_\alpha = \sqrt{z-\lambda_{\alpha,j}}\,\hat{p}_\alpha$ may give a nontrivial contribution to the integral. The last equality means that the condition $z-\lambda_{\alpha,j} = = z\sin^2\omega'_\alpha$ has to be satisfied for some $\omega'_\alpha \in [0, \pi/2]$. This condition is equivalent to the requirement $z\cos^2\omega_\alpha = \lambda_{\alpha,j}$ which may be obeyed for real $z \leq \lambda_{\alpha,j}$ only. However, by the definition of the surface $\Re$ such z do not belong to the sheets Π_l, $l_0 = \pm 1$ (see Sec. 5). Consequently, $\big(\big(\mathrm{J}^{(\alpha)}\Phi^* \mathrm{J}_0^\dagger\big)_j f\big)\big(\hat{k}_\alpha, z\big) = 0$ for any j and the first equality (6.10) holds. The second one may be proved analogously.

As already mentioned above, it follows from Eq. (6.10) that the last summand of (6.10) disappears and hence, Eq. (6.4) is true. This completes the proof of the theorem. □

Using Eqs. (6.3) and (6.4) one can present the Faddeev equations (2.7) continued into the sheet Π_l in the matrix form

$$M^l(z) \;=\; \mathbf{t}^l(z) \;-\; \mathbf{t}^l(z)\mathbf{R}_0^l(z)\Upsilon M^l(z) \tag{6.11}$$

where

$$\mathbf{t}^l(z) \;=\; \mathbf{t} \,-\, L_0 A_0 \mathbf{t} \mathbf{J}_0^\dagger \mathbf{s}_l^{-1} \mathbf{J}_0 \mathbf{t} \,-\, \Phi \mathrm{J}_1^\dagger L_1 A_1 \mathrm{J}_1 \Phi^*, \tag{6.12}$$

$$\mathbf{R}_0^l(z) \;=\; \mathbf{R}_0(z) \,+\, L_0 A_0(z) \mathbf{J}_0^\dagger(z) \mathbf{J}_0(z)\,. \tag{6.13}$$

Here, $\mathbf{s}_l(z) = \operatorname{diag}\{\mathbf{s}_{1,l}(z), \mathbf{s}_{2,l}(z), \mathbf{s}_{3,l}(z)\}$. By $M^l(z)$ we understand a supposed analytic continuation of the matrix $M(z)$ into the sheet Π_l.

Lemma 6.2. *For each two – body unphysical sheet Π_l of the surface $\Re$ there exists a path from the physical sheet Π_0 to the domain $\Pi_l^{(\mathrm{hol})}$ of Π_l which only passes trough two – body unphysical sheets $\Pi_{l'}$ and, moving on this path, the parameter z always stays in the respective domains $\Pi_{l'}^{(\mathrm{hol})} \subset \Pi_{l'}$.*

Proof. Let us make a use of the principle of mathematical induction. To this end, we rearrange the branching points $\lambda_{\alpha,j}$, $\alpha = 1,2,3,\ j = 1,2,\ldots,n_\alpha$, in nondecreasing order redenoting them as $\lambda_1, \lambda_2, \ldots, \lambda_m$, $m \le \sum_\alpha n_\alpha$, $\lambda_1 < \lambda_2 < < \cdots < \lambda_m$, and putting $\lambda_{m+1} = 0$. Let the multi–index $l = (l_1, l_2, \ldots, l_m)$ correspond temporarily namely to this enumeration. As previously, $l_j = 0$ if the sheet Π_l is related to the main branch of the function $(z-\lambda_j)^{1/2}$ otherwise $l_j = 1$. The index l_0 is omitted in these temporary notations.

It is clear that the transition of z from the physical sheet Π_0 through the segment (λ_1, λ_2) in the neighboring unphysical sheet $\Pi_{l^{(1)}}$ $\big(\text{into } \Pi_{l^{(1)}}^{\mathrm{hol}}\big)$, $l^{(1)} = \Big(l_1^{(1)}, l_2^{(1)}, \ldots, l_m^{(1)}\Big)$ with $l_1^{(1)} = 1$ and $l_j^{(1)} = 0$ for $j \ne 1$ is possible by definition of the domain $\Pi_{l^{(1)}}^{\mathrm{hol}}$ (see Sec. 4). According to Lemmas 4.3 and 4.4, if z belongs to $\Pi_{l^{(1)}}^{(\mathrm{hol})}$, it may be led to the real axis in the interval $\big(\lambda^{(1)}, +\infty\big)$ with certain $\lambda^{(1)} < \lambda_1$. Remaining in $\Pi_{l^{(1)}}^{(\mathrm{hol})}$, the point z may even go around the threshold λ_1 crossing the real axis in the segment $\big(\lambda^{(1)}, \lambda_1\big)$. Thus, the parameter z may be led from the sheet $\Pi_{l^{(1)}}$ into each neighboring unphysical sheet and, in particular, into the sheet Π_l identified by $l_1 = 0$, $l_2 = 1$, $l_j = 0$, $j \ge 3$. Transition of z from Π_0 through the segment (λ_2, λ_3) into the sheet Π_l with $l_1 = l_2 = 1$, $l_j = 0$, $j \ge 3$, is always possible.

We suppose further that the parameter z may be carried in this manner from Π_0 into all the two–body unphysical sheets $\Pi_{l^{(k)}}$ determined by the conditions $l_j^{(k)} = 0$, $j > k$. It is assumed also that during this motion z always remains in the domains $\Pi_{l^{(k)}}^{(\mathrm{hol})}$ of these sheets and does not visit other sheets. It follows from Lemmas 4.3 and 4.4 that if z stays in the domain $\Pi_{l^{(k)}}^{(\mathrm{hol})}$ of the sheet described, then it can be led to the real axis in the segment $\big(\lambda^{(k)}, +\infty\big)$ with a certain $\lambda^{(k)} < \lambda_k$. Hence, the parameter z from each of the sheets $\Pi_{l^{(k)}}$ may be carried through the interval $(\lambda_k, \lambda_{k+1})$ into the neighboring unphysical sheet $\Pi_{l^{(k+1)}}$ with $l_j^{(k+1)} = 1 - l_j^{(k)}$, $j \le k$, $l_{k+1}^{(k+1)} = 1$ and $l_j^{(k+1)} = 0$, $j > k+1$. This just means that z may be carried from Π_0 into all the two–body unphysical sheets $\Pi_{l^{(k+1)}}$ with $l_j^{(k+1)} = 0$, $j > k+1$. The whole time the parameter z remains in the holomorphy domains $\Pi_{l^{(k+1)}}^{(\mathrm{hol})}$ and does not visit the sheets $\Pi_{l^{(s)}}$ with $s > k+1$. By the principle of mathematical induction we conclude that the parameter z may be carried into all the two–body unphysical sheets.

The proof is completed. □

Using results of Sec. 4, as well as Lemma 6.2, we can prove the following important statement.

Theorem 6.3. *The iterations $\mathcal{Q}^{(n)}(z) = \bigl((-\mathbf{t}\mathbf{R}_0\Upsilon)^n\mathbf{t}\bigr)(z)$ for $n \geq 1$ of the absolute terms in the Faddeev equations (2.7) admit analytic continuation on the domain $\Pi_l^{(\mathrm{hol})}$ of each unphysical sheet $\Pi_l \subset \Re$ in the sense of distributions over $\mathcal{O}\bigl(\mathbb{C}^6\bigr)$. This continuation is described by the equalities $\mathcal{Q}^{(n)}(z)\big|_{\Pi_l} = \bigl((-\mathbf{t}^l\mathbf{R}_0^l\Upsilon)^n\mathbf{t}^l\bigr)(z)$.*

Remark 6.4. The products $L_1\mathrm{J}_1\Psi^*\Upsilon\mathcal{Q}^{(m)}$, $\mathcal{Q}^{(m)}\Upsilon\Psi\mathrm{J}_1^\dagger L_1$, $\tilde{L}_0\mathbf{J}_0\mathcal{Q}^{(m)}$, $\mathcal{Q}^{(m)}\mathbf{J}_0^\dagger\tilde{L}_0$, $L_1\mathrm{J}_1\Psi^*\Upsilon\mathcal{Q}^{(m)}\Upsilon\Psi\mathrm{J}_1^\dagger L_1$, $\tilde{L}_0\mathbf{J}_0\mathcal{Q}^{(m)}\mathbf{J}_0^\dagger\tilde{L}_0$, $L_1\mathrm{J}_1\Psi^*\Upsilon\mathcal{Q}^{(m)}\mathbf{J}_0^\dagger\tilde{L}_0$ and $\tilde{L}_0\mathbf{J}_0\mathcal{Q}^{(m)}\Upsilon\Psi\mathrm{J}_1^\dagger L_1$, $0 \leq m < n$, arising after substitution of the relations (6.12) and (6.13) into $\mathcal{Q}^{(n)}(z)\big|_{\Pi_l}$, have to be understood in the sense of the definitions from Sec. 4.

P r o o f. Theorem 6.3 will be proved in the case of the analytic continuation of the iteration $\mathcal{Q}^{(1)}(z)$. It will be clear from this proof that the iterations $\mathcal{Q}^{(n)}(z)$ with $n \geq 2$ could be considered in the same way as $\mathcal{Q}^{(1)}(z)$. We shall not expound here on the correspoding computations for $n \geq 2$, since they are too cumbersome.

So, let us consider the bilinear forms

$$\begin{aligned} Q_{\alpha\beta}(z) &= (f, \mathbf{t}_\alpha(z)R_0(z)\mathbf{t}_\beta(z)f') \\ &= \frac{1}{|s_{\alpha\beta}|}\int_{\mathrm{IR}^3} dk_\alpha \int_{\mathrm{IR}^3} dp_\alpha \int_{\mathrm{IR}^3} dk'_\beta \int_{\mathrm{IR}^3} dp'_\beta\, f(P)f'(P') \\ &\times \frac{t_\alpha\bigl(k_\alpha, \tilde{k}_\alpha^{(\beta)}(p_\alpha,p'_\beta), z-p_\alpha^2\bigr)t_\beta\bigl(\tilde{k}_\beta^{(\alpha)}(p_\alpha,p'_\beta), k'_\beta, z-p'^2_\beta\bigr)}{p_\alpha^2 + p'^2_\beta - 2c_{\alpha\beta}(p_\alpha,p'_\beta) - s_{\alpha\beta}^2 z} \end{aligned}$$

corresponding to the components $\mathcal{Q}^{(1)}_{\alpha\beta}(z) = -\mathbf{t}_\alpha(z)R_0(z)\mathbf{t}_\beta(z)$, $\beta \neq \alpha$, of the iteration $\mathcal{Q}^{(1)}(z)$, $\operatorname{Im} z \neq 0$. It is assumed that $f, f' \in \mathcal{O}\bigl(\mathbb{C}^6\bigr)$.

Using the spherical coordinates $p_\alpha \to \rho = |p_\alpha|^2$, $\hat{p}_\alpha$, $p'_\beta \to \rho' = |p'_\beta|^2$, $\hat{p}'_\beta$, in the integrals in the variables p_α and p'_β we get

$$\begin{aligned} &Q_{\alpha\beta}(z) \\ &= \frac{1}{4}\cdot\frac{1}{|s_{\alpha\beta}|}\int_{\mathrm{IR}^3} dk_\alpha \int_{\mathrm{IR}^3} dk'_\beta \int_{S^2} d\hat{p}_\alpha \int_{S^2} d\hat{p}'_\beta \int_0^\infty d\rho\sqrt{\rho}\int_0^\infty d\rho'\sqrt{\rho'} \\ &\times f\bigl(k_\alpha, \sqrt{\rho}\,\hat{p}_\alpha\bigr)\, f'\bigl(k'_\beta, \sqrt{\rho'}\,\hat{p}'_\beta\bigr) \\ &\times \frac{t_\alpha\bigl(k_\alpha, \tilde{k}_\alpha^{(\beta)}\bigl(\sqrt{\rho}\,\hat{p}_\alpha, \sqrt{\rho'}\,\hat{p}'_\beta\bigr), z-\rho\bigr)t_\beta\bigl(\tilde{k}_\beta^{(\alpha)}\bigl(\sqrt{\rho}\,\hat{p}_\alpha, \sqrt{\rho'}\,\hat{p}'_\beta\bigr), k'_\beta, z-\rho'\bigr)}{\rho + \rho - 2c_{\alpha\beta}\sqrt{\rho}\sqrt{\rho'}\bigl(\hat{p}_\alpha, \hat{p}'_\beta\bigr) - s_{\alpha\beta}^2 z}. \end{aligned} \tag{6.14}$$

Let us begin with continuation of the functions $Q_{\alpha\beta}(z)$ across the cut $(\lambda_{\min}, +\infty)$ to the left from the three – body threshold $\lambda_0 = 0$. We realize this continuation in the same way as the continuation of the kernels of $\mathbf{t}_\alpha(z)$ and $(\mathbf{t}_\alpha R_0)(z)$ in the proof of

Theorem 6.1. Additionally, we use the fact that for $z < 0$ the denominator of the expression under the integration sign in (6.14) cannot become equal to zero since

$$\rho+\rho' - 2c_{\alpha\beta}\sqrt{\rho}\sqrt{\rho'}\big(\hat{p}_\alpha,\hat{p}'_\beta\big) \;\geq\; (1-|c|_{\alpha\beta})(\rho+\rho') \quad \text{for all} \quad \rho,\,\rho' > 0\,,\;\; \hat{p}_\alpha,\,\hat{p}'_\beta \in S^2\,.$$

Thus, when continuing across the segment $(\lambda_{\min},0)$, only residues at the poles $\lambda_{\alpha,j}$, $\lambda_{\beta,k}$ give a nontrivial contribution to the Cauchy type integrals in the variables ρ, ρ' generated in (6.14) by the two−body singularities (the poles of $g_{\alpha,j}(z)$ [see Eq. (2.13)]).

Let us continue the function (6.14) across the segment $(\lambda^{(1)},\lambda^{(2)})$ where $\lambda^{(1)},\lambda^{(2)}$, $\lambda^{(1)}<\lambda^{(2)}$, are some neighboring points of the set $\sigma_d^{(2)}=\bigcup_{\alpha=1}^3\sigma_d(h_\alpha)$, $\big(\lambda^{(1)},\lambda^{(2)}\big)\cap$ $\cap\sigma_d^{(2)}=\emptyset$. Then the totality of the kernels $\mathbf{t}_\alpha(z)R_0(z)\mathbf{t}_\beta(z)$, $\alpha,\,\beta=1,2,3$, $\beta\neq\alpha$, may be continued in z into the two−body unphysical sheet $\Pi_l\subset\Re$ neighboring the physical one and such that its indices $l_0=0$, $l_{\gamma,j}=1$ if $\lambda_{\gamma,j}>\lambda^{(1)}$, and $l_{\gamma,j}=0$ if $l_{\gamma,j}\geq\lambda^{(2)}$, $\gamma=1,2,3$, $j=1,2,\ldots,n_\gamma$. One can easily check that the continuation of the functions $Q_{\alpha\beta}(z)$ into these sheets $\big($in a vicinity of the segment $\big(\lambda^{(1)},\lambda^{(2)}\big)\big)$ corresponds exactly to the iteration $\mathbf{t}^l(z)\mathbf{R}_0(z)\Upsilon\mathbf{t}^l(z)$ of the absolute term of the continued Faddeev equations (6.11). Recall that in the two−body sheets $\mathbf{t}^l(z)=\mathbf{t}(z)-\Phi\mathrm{J}_1^\dagger L_1A_1\mathrm{J}_1\Phi^*$. Hence the analyticity domain of the kernels $\mathbf{t}^l(z)\mathbf{R}_0(z)\Upsilon\mathbf{t}^l(z)$ in these sheets is determined by the set of those points where the functions

$$\begin{aligned}
F_{\alpha,j;\,\beta,k}(z,\eta) &= \big(z-\lambda_{\alpha,j}+z-\lambda_{\beta,k}-2c_{\alpha\beta}\sqrt{z-\lambda_{\alpha,j}}\,\sqrt{z-\lambda_{\beta,j}}\,\eta-s^2_{\alpha\beta}z\big)^{-1}\,,\\
F_{\alpha,j}(z,\rho',\eta) &= \big(z-\lambda_{\alpha,j}+\rho'-2c_{\alpha\beta}\sqrt{z-\lambda_{\alpha,j}}\,\sqrt{\rho'}\,\eta-s^2_{\alpha\beta}z\big)^{-1}\,,\\
F_{\beta,k}(z,\rho,\eta) &= \big(\rho+z-\lambda_{\beta,k}-2c_{\alpha\beta}\sqrt{\rho}\,\sqrt{z-\lambda_{\beta,j}}\,\eta-s^2_{\alpha\beta}z\big)^{-1}\,,
\end{aligned}$$

are holomorphic in z for all $\rho,\,\rho'>0$, $\eta\in[-1,1]$. The latter arise in (6.14) due to the presence of the factors (4.25) as a result of taking residues at the poles $\rho=z-\lambda_{\alpha,j}$ and/or $\rho'=z-\lambda_{\beta,k}$, $\lambda_{\alpha,j}$, $\lambda_{\beta,k}\leq\lambda^{(1)}$, $\alpha,\,\beta=1,2,3$, $\alpha\neq\beta$. The domains where the singularities of the above functions are situated have been described in Lemmas 4.3 and 4.4. It follows from these lemmas that the product $\mathbf{t}^l(z)\mathbf{R}_0(z)\Upsilon\mathbf{t}^l(z)$ describes the analytic continuation of the iteration $\mathcal{Q}^{(1)}(z)$ on the domain $\Pi_l^{(\mathrm{hol})}$ of each neighboring (with respect to Π_0) two−body sheet Π_l. Note that the singularities of the functions $F_{\alpha,j;\,\beta,k}(z,\eta)$, $F_{\alpha,j}(z,\rho',\eta)$ and $F_{\beta,k}(z,\rho,\eta)$ are, as a matter of fact, three−body ones though being situated in the two−body unphysical sheets. Indeed, making the substitution $\eta=(\hat{p}_\alpha,\hat{p}'_\beta)$ in (6.14) one finds that the integral in the variable η turns out to be a Cauchy type integral. This means that, e. g., the points $z_{\mathrm{rl}}, z_{\mathrm{rt}}$ (see Lemma 4.4) are extra logarithmic branching points. After crossing the cuts on Π_l along the segments $[z_{\mathrm{rl}},z_{\mathrm{rt}}]$ as well as the root ellipses from Lemma 4.4, the representation of the analytic continuation of $\mathcal{Q}^{(1)}(z)$ in the form of the product $\mathbf{t}^l\mathbf{R}_0^l\mathbf{t}^l$ becomes invalid. In the present paper we restrict ourselves to considering only those domains of the unphysical sheets where the correctness of such representations is not violated.

Let us now use Lemma 6.2 and carry out a continuation of the form $Q_{\alpha\beta}(z)$ into the rest of the two−body unphysical sheets. Boundaries of the holomorphy domains $\Pi_l^{(\mathrm{hol})}$ of $Q_{\alpha\beta}(z)$ of these sheets are determined again only by the indices α, j and β, k of the functions $F_{\alpha,j;\beta,k}$, $F_{\alpha,j}$ and $F_{\beta,k}$ included in the kernels of the operators

$$L_1\mathrm{J}_1\Phi^*\mathbf{R}_0\Upsilon\Phi\mathrm{J}_1^\dagger L_1 \;\equiv\; L_1\mathrm{J}_1\Psi^*\Upsilon\mathbf{v}\Psi\mathrm{J}_1^\dagger L_1\,,$$

$$L_1 \mathrm{J}_1 \Phi^* \mathbf{R}_0 \Upsilon \mathbf{t} \;\equiv\; L_1 \mathrm{J}_1 \Psi^* \Upsilon \mathbf{t}\,,$$
$$\mathbf{t} \mathbf{R}_0 \Upsilon \Phi \mathrm{J}_1^\dagger L_1 \;\equiv\; \mathbf{t} \Upsilon \Psi \mathrm{J}_1^\dagger L_1$$

arising in the product $\mathbf{t}^l \mathbf{R}_0^l \Upsilon \mathbf{t}^l$.

Thus we can state that the kernels of the iteration $\mathcal{Q}^{(1)}(z)$ admit an immediate analytic continuation as holomorphic generalized functions over $\mathcal{O}(\mathbb{C}^6)$ on the domains $\Pi_l^{(\mathrm{hol})}$ of all the two – body unphysical sheets where

$$\mathcal{Q}^{(1)}(z)\big|_{\Pi_l} \;=\; \mathbf{t}^l(z) \mathbf{R}_0(z) \Upsilon \mathbf{t}^l(z)\,.$$

Let us consider now the continuation of the iteration $\mathcal{Q}^{(1)}(z)$ into the three – body unphysical sheets Π_l with $l_0 = \pm 1$. It is clear that the two – body singularities will give the same contribution to the continued kernels as before when continuing this iteration into the two – body sheets. Therefore we assume here for the sake of simplicity that these singularities are absent or, in other words, that the two – body subsystems have no discrete spectrum.

Let us deal, say, with the continuation of $\mathcal{Q}^{(1)}(z)$ into the sheet Π_l with $l_0 = +1$. This means that we study a crossing of the ray $(0, +\infty)$ from below going upward. We begin with taking in (6.14) the limit $z \to E - i0$, $E > 0$, and rewriting the limit values of the T – matrix t_α (t_β) at the points $E - \rho - i0$ ($E - \rho' - i0$) of the segment $(0, E)$ in terms of the continued kernel $t_\alpha\big(k_\alpha, k'_\alpha, z\big)$ $\big(t_\beta\big(k_\beta, k'_\beta, z\big)\big)$ on the unphysical sheet using the representation (3.11). For t_α we have

$$\begin{aligned}
& t_\alpha\big(k_\alpha, \tilde{k}_\alpha^{(\beta)}\big(\sqrt{\rho}\,\hat{p}_\alpha, \sqrt{\rho'}\,\hat{p}'_\beta\big), E - \rho - i0\big) \\
= \;& t_\alpha\big(k_\alpha, \tilde{k}_\alpha^{(\beta)}\big(\sqrt{\rho}\,\hat{p}_\alpha, \sqrt{\rho'}\,\hat{p}'_\beta\big), E - \rho + i0\big) \\
& +\; \pi i \sqrt{\rho + i0}\, \tau_\alpha\big(k_\alpha, \tilde{k}_\alpha^{(\beta)}\big(\sqrt{\rho}\,\hat{p}_\alpha, \sqrt{\rho'}\,\hat{p}'_\beta\big), E - \rho + i0\big)
\end{aligned}$$

and analogously for t_β. At $\rho > E$ ($\rho' > E$) the limit values of the T – matrix $t_\alpha(\,\ldots, E - \rho - i0)$ $\big(t_\beta\big(\,\ldots, E - \rho - i0\big)\big)$ from below coincide with the limit values $t_\alpha(\,\ldots, E - \rho + i0)$ $\big(t_\beta\big(\,\ldots, E - \rho + i0\big)\big)$ from above, in view of analyticity of $t_\alpha(z)$ ($t_\beta(z)$) in z at $z \notin \mathbb{R}^+$. In the same way we rewrite as well the denominator of the expression under the integration sign in (6.14),

$$\begin{aligned}
\frac{1}{\mathrm{F}\big(\rho, \rho', \hat{p}_\alpha, \hat{p}'_\beta\big) - s_{\alpha\beta}^2 + i0} \;=\; & \frac{1}{\mathrm{F}\big(\rho, \rho', \hat{p}_\alpha, \hat{p}'_\beta\big) - s_{\alpha\beta}^2 - i0} \\
& -\; 2\pi i \delta\big(\mathrm{F}\big(\rho, \rho', \hat{p}_\alpha, \hat{p}'_\beta\big) - s_{\alpha\beta}^2 E\big)
\end{aligned}$$

where $\mathrm{F} = \rho + \rho' - 2c_{\alpha\beta}\sqrt{\rho}\,\sqrt{\rho'}\big(\hat{p}_\alpha, \hat{p}'_\beta\big)$.

Note right away that the kernel $\delta(\mathrm{F}(\rho, \rho', \hat{p}_\alpha, \hat{p}'_\beta) - s_{\alpha\beta}^2 E)$ corresponds to the distribution $\big(\mathrm{J}_0^\dagger \mathrm{J}_0\big)(P, P', E \pm i0)$. Also, it is easy to find that all the terms of (6.14) including the δ – function $\delta\big(\mathrm{F}\big(\rho, \rho', \hat{p}_\alpha, \hat{p}'_\beta\big) - s_{\alpha\beta}^2 E\big)$ generate as a sum a bilinear form corresponding to the kernel $l_0 A_0(E)\big(\mathbf{t}_\alpha^l \mathrm{J}_0^\dagger \mathrm{J}_0 \mathbf{t}_\beta^l\big)(P, P', E + i0)$ admitting analytic continuation on $\Pi_l \cap \mathcal{P}_b$, $l_0 = 1$ in the sense of distributions over $\mathcal{O}(\mathbb{C}^6)$.

Then we consider the terms of (6.14) including the factor $1/\big(\mathrm{F}\big(\rho,\rho',\hat p_\alpha,\hat p'_\beta\big)-s^2_{\alpha\beta}E-$ $-i0\big)$. The simplest of the summands includes the fraction

$$\frac{1}{|s_{\alpha\beta}|}\cdot\frac{t_\alpha(\dots,E-\rho+i0)t_\beta(\dots,E-\rho'+i0)}{\rho+\rho'-2c_{\alpha\beta}\sqrt{\rho}\,\sqrt{\rho'}\big(\hat p_\alpha,\hat p'_\beta\big)-s^2_{\alpha\beta}E-i0}\,.$$

In all this, the integration in ρ as well as in ρ' is carried out along the interval $(0,+\infty)$. Evidently, this summand represents a boundary value at $z=E+i0$ of the bilinear form for the product $\mathbf{t}_\alpha(z)R_0(z)\mathbf{t}_\beta(z)$.

We consider contributions of the summands which include the products $\tau_\alpha(\dots)t_\beta(\dots)$ and $t_\alpha(\dots)\tau_\beta(\dots)$ for the case of the first of such summands.

Let us rewrite the respective bilinear form,

$$\begin{aligned}
Q^{(t\tau)}_{\alpha\beta}(E) \;&=\; \frac14\cdot\frac{1}{|s_{\alpha\beta}|}\int_{\mathbb{R}^3}dk_\alpha\int_{\mathbb{R}^3}dk'_\beta\int_{S^2}d\hat p_\alpha\int_{S^2}d\hat p'_\beta\int_0^E d\rho\,\sqrt{\rho}\int_0^\infty d\rho'\,\sqrt{\rho'}\\
&\times\;\pi i\,\sqrt{\rho}f\big(k_\alpha,\sqrt{\rho}\,\hat p_\alpha\big)\,f'\big(k'_\beta,\sqrt{\rho'}\hat p'_\beta\big)\\
&\times\;\frac{\tau_\alpha\big(k_\alpha,\tilde k^{(\beta)}_\alpha\big(\sqrt{\rho}\,\hat p_\alpha,\sqrt{\rho'}\,\hat p'_\beta\big),E-\rho+i0\big)}{\rho+\rho-2c_{\alpha\beta}\,\sqrt{\rho}\,\sqrt{\rho'}\big(\hat p_\alpha,\hat p'_\beta\big)-s^2_{\alpha\beta}E-i0}\,.\\
&\times\;\frac{t_\beta\big(\tilde k^{(\alpha)}_\beta\big(\sqrt{\rho}\,\hat p_\alpha,\sqrt{\rho'}\,\hat p'_\beta\big),k'_\beta,E-\rho'+i0\big)}{\rho+\rho-2c_{\alpha\beta}\,\sqrt{\rho}\,\sqrt{\rho'}\big(\hat p_\alpha,\hat p'_\beta\big)-s^2_{\alpha\beta}E-i0}
\end{aligned}\tag{6.15}$$

We use Lemma 4.6 to prove the existence of an analytic continuation of the function $Q^{(t\tau)}_{\alpha\beta}(E)$ in the domain $\operatorname{Im}z>0$. To apply this lemma we divide the interval of integration in the variable ρ' in (6.15) into two intervals $[0,E]$ and $(E,+\infty)$. Then we get in (6.15) two terms including $\dots\int_0^E d\rho\int_0^E d\rho'\,\dots$ and $\dots\int_0^E d\rho\int_E^{+\infty}d\rho'\,\dots$. In the first term we make two changes of variables, $\rho\to\nu$, $\rho=\nu E$, and $\rho'\to\nu'$, $\rho'=\nu'E$, and in the second term, only the first change, $\rho=\nu E$. As a result we find that

$$Q^{(t\tau)}_{\alpha\beta}(E)\;=\;Q^{(1)}_{\alpha\beta}(E+i0)\;+\;Q^{(2)}_{\alpha\beta}(E+i0)$$

where by $Q^{(1)}_{\alpha\beta}(E+i0)$ and $Q^{(2)}_{\alpha\beta}(E+i0)$ we understand the boundary values (at $z=E+i0$, $E>0$) of the functions

$$\begin{aligned}
Q^{(1)}_{\alpha\beta}(z)\;&=\;\frac14\cdot\frac{(\sqrt{z})^5}{|s_{\alpha\beta}|}\int_{\mathbb{R}^3}dk_\alpha\int_{\mathbb{R}^3}dk'_\beta\int_{S^2}d\hat p_\alpha\int_{S^2}d\hat p'_\beta\int_0^1 d\nu\sqrt{\nu}\int_0^1 d\nu'\sqrt{\nu'}\\
&\times\;f\big(k_\alpha,\sqrt{z}\,\sqrt{\nu}\,\hat p_\alpha\big)\cdot f'\big(k'_\beta,\sqrt{z}\sqrt{\nu'}\hat p'_\beta\big)\\
&\times\;\frac{\pi i\,\sqrt{\nu}\,\tau_\alpha(\dots,z(1-\nu))\cdot t_\beta(\dots,z(1-\nu'))}{\nu+\nu-2c_{\alpha\beta}\,\sqrt{\nu}\,\sqrt{\nu'}\big(\hat p_\alpha,\hat p'_\beta\big)-s^2_{\alpha\beta}-i0}\,,
\end{aligned}$$

and

$$Q^{(2)}_{\alpha\beta}(z) = \frac{1}{4}\cdot\frac{z^2}{|s_{\alpha\beta}|}\int_{\mathrm{I\!R}^3} dk_\alpha \int_{\mathrm{I\!R}^3} dk'_\beta \int_{S^2} d\hat p_\alpha \int_{S^2} d\hat p'_\beta \int_0^1 d\nu\,\sqrt{\nu}\int_{\Gamma_z} d\rho'\,\sqrt{\rho'}$$
$$\times\; f\big(k_\alpha,\sqrt{z}\,\sqrt{\nu}\,\hat p_\alpha\big)\cdot f'\big(k'_\beta,\sqrt{\rho'}\,\hat p'_\beta\big) \tag{6.16}$$
$$\times\; \frac{\pi i\,\sqrt{\nu}\,\tau_\alpha(\ldots,z(1-\nu))\cdot t_\beta(\ldots,z-\rho')}{\nu z+\rho'-2c_{\alpha\beta}\sqrt{\nu}\,\sqrt{z}\,\sqrt{\rho'}\big(\hat p_\alpha,\hat p'_\beta\big)-s^2_{\alpha\beta}z},$$

respectively. For Γ_z we take the same path as in (4.52).

By the same reasoning as in the consideration of the forms (4.51) and (4.52), we conclude that the functions $Q^{(1)}_{\alpha\beta}(z)$ and $Q^{(2)}_{\alpha\beta}(z)$ admit continuations in the domain $\operatorname{Im} z>0$.

Therefore, we have proved that the function $Q^{(t\tau)}_{\alpha\beta}(E)$ admits an analytic continuation on the domain $\Pi^{(\mathrm{hol})}_l$ of the sheet Π_l, $l_0=+1$. Analogously, an analytic continuation on $\Pi^{(\mathrm{hol})}_l$, $l_0=+1$ exists as well for the bilinear form $Q^{(\tau t)}_{\alpha\beta}(E)$ corresponding to the contribution in (6.14) from the product $t_\alpha(\ldots)\tau_\beta(\ldots)$.

To this end, let us consider the contribution of the product $\tau_\alpha(\ldots)\tau_\beta(\ldots)$. The respective bilinear form $Q^{(\tau\tau)}_{\alpha\beta}(E)$ reads

$$Q^{(\tau\tau)}_{\alpha\beta}(E) = \frac{1}{4}\cdot\frac{-\pi^2}{|s_{\alpha\beta}|}\int_{\mathrm{I\!R}^3} dk_\alpha \int_{\mathrm{I\!R}^3} dk'_\beta \int_{S^2} d\hat p_\alpha \int_{S^2} d\hat p'_\beta \int_0^E d\rho\rho\int_0^E d\rho'\rho'$$
$$\times\; f\big(k_\alpha,\sqrt{\rho}\,\hat p_\alpha\big)f'\big(k'_\beta,\sqrt{\rho'}\,\hat p'_\beta\big)$$
$$\times\; \frac{\tau_\alpha\big(k_\alpha,\tilde k^{(\beta)}_\alpha\big(\sqrt{\rho}\,\hat p_{alpha},\sqrt{\rho'}\hat p'_\beta\big),E-\rho+i0\big)}{\rho+\rho'-2c_{\alpha\beta}\sqrt{\rho}\,\sqrt{\rho'}\big(\hat p_\alpha,\hat p'_\beta\big)-s^2_{\alpha\beta}E-i0} \tag{6.17}$$
$$\times\; \frac{\tau_\beta\big(\tilde k^{(\alpha)}_\beta\big(\sqrt{\rho}\,\hat p_\alpha,\sqrt{\rho'}\,\hat p'_\beta\big),k'_\beta,E-\rho'+i0\big)}{\rho+\rho'-2c_{\alpha\beta}\sqrt{\rho}\,\sqrt{\rho'}\big(\hat p_\alpha,\hat p'_\beta\big)-s^2_{\alpha\beta}E-i0}.$$

Making the change of variables $\rho\to\nu$, $\rho'\to\nu'$, $\rho=E\nu$, $\rho'=E\nu'$, we get the integral

$$Q^{(\tau\tau)}_{\alpha\beta}(z) = \frac{-\pi^2}{|s_{\alpha\beta}|}\cdot\frac{z^3}{4}\int_{\mathrm{I\!R}^3} dk_\alpha \int_{\mathrm{I\!R}^3} dk'_\beta \int_{S^2} d\hat p_\alpha \int_{S^2} d\hat p'_\beta \int_0^1 d\nu\nu\int_0^1 d\nu'\nu'$$
$$\times\; f\big(k_\alpha,\sqrt{z}\,\sqrt{\nu}\,\hat p_\alpha\big)\,f'\big(k'_\beta,\sqrt{z}\,\sqrt{\nu'}\hat p'_\beta\big)$$
$$\times\; \frac{\tau_\alpha(\ldots,z(1-\nu))\tau_\beta(\ldots,z(1-\nu'))}{\nu+\nu'-2c_{\alpha\beta}\sqrt{\nu}\,\sqrt{\nu'}\big(\hat p_\alpha,\hat p'_\beta\big)-s^2_{\alpha\beta}-i0},$$

where the denominator of the expression under the integral sign includes no dependence on the parameter z. In view of holomorphy in z of the numerator of this expression, the integral $Q^{(\tau\tau)}_{\alpha\beta}(z)$ admits an immediate analytic continuation on $\Pi^{(\mathrm{hol})}_l$, $l=+1$.

Summarizing the above, we can assert that the kernels of the iteration $\mathcal{Q}^{(1)}(z)$ admit analytic continuation (in the sense of distributions over $\mathcal{O}(\mathbb{C}^6)$) on the domain $\Pi_l^{(\mathrm{hol})}$ of the three-body unphysical sheet Π_l, $l_0 = +1$. A similar assertion holds as well for the three-body sheet Π_l with $l_0 = -1$. Also, we can state that the result of continuation may be represented as

$$\begin{aligned}\mathcal{Q}^{(1)}\big|_{\Pi_l} &= \mathbf{t}^l\mathbf{R}_0^l\Upsilon\mathbf{t}^l \\ &= \big(\mathbf{t} - L_0A_0\mathbf{t}\mathbf{J}_0^\dagger\mathbf{s}_l^{-1}\mathbf{J}_0\mathbf{t}\big)\big(\mathbf{R}_0 + L_0A_0\mathbf{J}_0^\dagger\mathbf{J}_0\big)\Upsilon\big(\mathbf{t} - L_0A_0\mathbf{t}\mathbf{J}_0^\dagger\mathbf{s}_l^{-1}\mathbf{J}_0\mathbf{t}\big)\,.\end{aligned} \tag{6.18}$$

When studying a continuation of the iteration $\mathcal{Q}^{(1)}(z)$ into the three-body unphysical sheets Π_l, $l_0 = \pm 1$ in the general case where the pair subsystems may have eigenstates one arrives again at the formula $\mathcal{Q}^{(1)}\big|_{\Pi_l} = \mathbf{t}^l\mathbf{R}_0^l\Upsilon\mathbf{t}^l$. However, in contrast to (6.18) one must now use for $\mathbf{t}^l(z)$ the total expressions (6.12).

The proof is completed. □

Remark 6.5. Theorem 6.3 means that one can pose the continued Faddeev equations (6.11) only in the domains $\Pi_l^{(\mathrm{hol})} \subset \Pi_l$.

7. Representations for the analytic continuation of the matrix $M(z)$ in unphysical sheets

In the present section we use the continued Faddeev equations (6.11) to obtain representations for the matrix $M^l(z)$ in the domains $\Pi_l^{(\mathrm{hol})}$ of the unphysical sheets $\Pi_l \subset \Re$. The representations will be given in terms of the matrix $M(z)$ components themselves taken in the physical sheet or, more precisely, in terms of the half-on-shell matrix $M(z)$ as well as the inverse operators of the truncated scattering matrices $S_l(z)$ and $S_l^\dagger(z)$. As a matter of fact, the construction of the representations for $M^l(z)$ consists in explicitly "solving" the continued Faddeev equations (6.11) in the same way as in [61], [62] where representations of the type (3.11) had been found for analytic continuation of the T-matrix in the multichannel scattering problem with binary channels. We consider derivation of the representations for $M^l(z)$ as a constructive proof of existence (in the sense of distributions over $\times_{\alpha=1}^{3}\mathcal{O}(\mathbb{C}^6)$) of analytic continuation of the matrix $M(z)$ into the unphysical sheets Π_l of the surface $\Re$.

So, let us consider the Faddeev equations (6.11) in the sheet Π_l with $l_0 = 0$ or $l_0 = \pm 1$ and $l_{\beta,j} = 0$ or $l_{\beta,j} = 1$, $\beta = 1, 2, 3$, $j = 1, 2, \ldots, n_\beta$. Using the expressions (6.12) for $\mathbf{t}^l(z)$ and (6.13) for $\mathbf{R}_0^l(z)$, we transfer all the summands including $M^l(z)$ but not $\mathbf{J}_0$ and J_1, to the left side of Eqs. (6.11). Making then a simple transformation based on the identity $\mathbf{s}_l^{-1}(z) = \hat{\mathbf{I}}_0 - \mathbf{s}_l^{-1}(z)\mathbf{J}_0(z)\mathbf{t}(z)\mathbf{J}_0^\dagger(z)A_0(z)L_0$ we rewrite (6.11) in the form

$$(\mathbf{I} + \mathbf{t}\mathbf{R}_0\Upsilon)M^l = \mathbf{t}\Big[\mathbf{I} - A_0^{(l)}\mathbf{J}_0^\dagger\mathbf{s}_l^{-1}\mathbf{J}_0\mathbf{t} - A_0^{(l)}\mathbf{J}_0\mathbf{X}_0^{(l)}\Big] - \Phi\mathrm{J}_1^\dagger A_1^{(l)}\Big(\mathrm{J}_1\Phi^* + \mathbf{X}_1^{(l)}\Big) \tag{7.1}$$

where $A_0^{(l)}(z)=L_0A_0(z)$, $A_1^{(l)}(z)=L_1A_1(z)$. Besides, we denote

$$\begin{aligned}\mathbf{X}_0^{(l)} &= |L_0|\mathbf{s}_l^{-1}\mathbf{J}_0(\mathbf{I}-\mathbf{t}\mathbf{R}_0)\Upsilon M^l\,,\\ \mathbf{X}_1^{(l)} &= -L_1\Big[\mathrm{J}_1\Phi^*\mathbf{R}_0+A_0^{(l)}\mathrm{J}_1\Phi^*\mathbf{J}_0^\dagger\mathbf{J}_0\Big]\Upsilon M^l\,.\end{aligned} \tag{7.2}$$

It should be noted that

$$\mathrm{J}_1\Phi^*\mathbf{R}_0 \;=\; -\,\mathrm{J}_1\Psi^*\,. \tag{7.3}$$

Indeed,

$$\begin{aligned}\big(\mathrm{J}_{\alpha,j}\langle\,\cdot\,,\phi_{\alpha,j}\rangle R_0\big)\big(\hat p_\alpha,P'\big) &= \int_{\mathbb{R}^3}dk''_\alpha\int_{\mathbb{R}^3}dp''_\alpha\frac{\delta\big(\sqrt{z-\lambda_{\alpha,j}}-|p''_\alpha|\big)}{|p''_\alpha|^2}\\ &\quad\times\;\delta\big(\hat p_\alpha,\hat p''_\alpha\big)\,\overline{\phi_{\alpha,j}}\big(k''_\alpha\big)\ldots\frac{\delta\big(k''_\alpha-k'_\alpha\big)\delta\big(p''_\alpha-p'_\alpha\big)}{{k'}^2_\alpha+{p'}^2_\alpha-z}\\ &= \frac{\overline{\phi_{\alpha,j}}(k'_\alpha)\delta\big(\sqrt{z-\lambda_{\alpha,j}}-|p'_\alpha|\big)}{|p'_\alpha|^2\,({k'_\alpha}^2+z-\lambda_{\alpha,j}-z)}\cdot\delta\big(\hat p_\alpha,\hat p'_\alpha\big)\\ &= \frac{\overline{\phi_{\alpha,j}}(k'_\alpha)}{{k'_\alpha}^2-\lambda_{\alpha,j}}\,\mathrm{J}_{\alpha,j}\big(z,\hat p_\alpha,p'_\alpha\big)\,.\end{aligned}$$

Then it follows from Eq. (3.13) that $\mathrm{J}_{\alpha,j}\langle\,\cdot\,,\phi_{\alpha,j}\rangle R_0=-\mathrm{J}_{\alpha,j}\langle\,\cdot\,,\psi_{\alpha,j}\rangle$, and thereby the equality (7.3) is really true.

Along with (7.3) the equalities

$$\Big(\mathrm{J}_1\Phi^*\mathbf{J}_0^\dagger\Big)(z) \;=\; 0\,,\quad \Big(\mathbf{J}_0\Phi\mathrm{J}_1^\dagger\Big)(z) \;=\; 0\,, \tag{7.4}$$

hold in accordance with (6.10) for all $z\in\mathbb{C}\setminus(-\infty,\lambda_{\max}]$.

Note that the condition $z\notin(-\infty,\lambda_{\max})$ necessary for Eq. (7.4) to be valid, does not apply to the two−body unphysical sheets Π_l, $l_0=0$, since in these sheets $A_0^{(l)}(z)=0$ and consequently, the terms including the products $\mathbf{J}_0^\dagger\mathbf{J}_0$ are absent in (7.1). Meanwhile, the points $z\in(-\infty,\lambda_{\max}]$ were excluded from the three−body sheets Π_l, $l_0=\pm1$, by definition.

Using Eq. (7.3) and the first of Eqs. (7.4) one can rewrite $\mathbf{X}_1^{(l)}$ in the form

$$\mathbf{X}_1^{(l)} \;=\; L_1\mathrm{J}_1\Psi^*\Upsilon M^l\,. \tag{7.5}$$

Notice further that the operator $\mathbf{I}+\mathbf{t}\mathbf{R}_0\Upsilon$ admits an explicit inversion in terms of $M(z)$,

$$(\mathbf{I}+\mathbf{t}\mathbf{R}_0\Upsilon)^{-1} \;=\; \mathbf{I}\;-\;M\Upsilon\mathbf{R}_0\,, \tag{7.6}$$

for all $z\in\Pi_0$ which do not belong to the discrete spectrum $\sigma_d(H)$ of the Hamiltonian H, and

$$(\mathbf{I}-M\Upsilon\mathbf{R}_0)\mathbf{t} \;=\; M\,. \tag{7.7}$$

The equality (7.6) is a simple consequence of the Faddeev equations (2.7) and the identity $\mathbf{R}_0\Upsilon = \Upsilon\mathbf{R}_0$. The relation (7.7) represents the alternative variant (2.8) of these equations. Now, we can rewrite Eqs. (7.1) in equivalent form

$$
\begin{aligned}
M^l \;&=\; M\Big(\mathbf{I} - A_0^{(l)}\mathbf{J}_0^\dagger\mathbf{s}_l^{-1}\mathbf{J}_0\mathbf{t} - A_0^{(l)}\mathbf{J}_0^\dagger\mathbf{X}_0^{(l)}\Big)\\
&-\;(\mathbf{I} - M\Upsilon\mathbf{R}_0)\Phi\mathrm{J}_1^\dagger A_1^{(l)}\Big(\mathrm{J}_1\Phi^* + \mathbf{X}_1^{(l)}\Big).
\end{aligned} \tag{7.8}
$$

Eq. (7.8) means that the matrix $M^l(z)$ is expressed in terms of the quantities $\mathbf{X}_0^{(l)}(z)$ and $\mathbf{X}_1^{(l)}(z)$. The main goal of this section consists in the representation of these quantities in terms of the matrix $M(z)$ considered in the physical sheet.

To obtain for $\mathbf{X}_0^{(l)}$ and $\mathbf{X}_1^{(l)}$ a closed system of equations we use the definitions (7.2) and (7.5) and apply the operators $\mathbf{s}_l^{-1}\mathbf{J}_0(\mathbf{I}-\mathbf{t}\mathbf{R}_0)\Upsilon$ and $\mathrm{J}_1\Psi^*$ to both parts of Eq. (7.8). At the moment we use also the identities

$$
[\mathbf{I} - \mathbf{t}\mathbf{R}_0]\Upsilon M \;=\; M_0 \,-\, \mathbf{t}\,, \quad [\mathbf{I} - \mathbf{t}\mathbf{R}_0]\Upsilon[\mathbf{I} - M\Upsilon\mathbf{R}_0] \;=\; [\mathbf{I} - M_0\mathbf{R}_0]\Upsilon \tag{7.9}
$$

where $M_0 = \Omega^\dagger\Omega M = (\mathbf{I} + \Upsilon)M$. The relations (7.9) are another easily verified consequence of the Faddeev equations (2.7). Along with Eqs. (7.9) we use here also the second of the equalities (7.4). As a result we come to the desired system of equations for $\mathbf{X}_0^{(l)}$ and $\mathbf{X}_1^{(l)}$:

$$
\begin{aligned}
\mathbf{X}_0^{(l)} \;&=\; |L_0|\,\mathbf{s}_l^{-1}\mathbf{J}_0\Big[(M_0 - \mathbf{t})\Big(\mathbf{I} - A_0^{(l)}\mathbf{J}_0^\dagger\mathbf{s}_l^{-1}\mathbf{J}_0\mathbf{t} - A_0^{(l)}\mathbf{J}_0^\dagger\mathbf{X}_0^{(l)}\Big)\Big]\\
&- |L_0|\,\mathbf{s}_l^{-1}\mathbf{J}_0 M_0\Upsilon\Psi\mathrm{J}_1^\dagger A_1^{(l)}\Big(\mathrm{J}_1\Phi^* + \mathbf{X}_1^{(l)}\Big),
\end{aligned} \tag{7.10}
$$

$$
\begin{aligned}
\mathbf{X}_1^{(l)} \;&=\; L_1\mathrm{J}_1\Psi^*\Upsilon M\Big(\mathbf{I} - A_0^{(l)}\mathbf{J}_0^\dagger\mathbf{s}_l^{-1}\mathbf{J}_0^\dagger\mathbf{t} - A_0^{(l)}\mathbf{J}_0^\dagger\mathbf{X}_0^{(l)}\Big)\\
&-\; L_1\mathrm{J}_1\Psi^*\Upsilon[\Phi + M\Upsilon\Psi]\mathrm{J}_1^\dagger A_1^{(l)}\Big(\mathrm{J}_1\Phi^* + \mathbf{X}_1^{(l)}\Big).
\end{aligned} \tag{7.11}
$$

It is convenient to rewrite this system in matrix form

$$
\widetilde{B}^{(l)}\mathbf{X}^{(l)} \;=\; \widetilde{D}^{(l)}\,, \quad \mathbf{X}^{(l)} \;=\; \Big(\mathbf{X}_0^{(l)}, \mathbf{X}_1^{(l)}\Big)^\dagger
$$

with $\widetilde{B}^{(l)} = \big\{\widetilde{B}_{ij}^{(l)}\big\},\ i, j = 0, 1$, the matrix consisting of the operators appearing in the unknowns $\mathbf{X}_0^{(l)}$ and $\mathbf{X}_1^{(l)}$. By $\widetilde{D}^{(l)}$, $\widetilde{D}^{(l)} = \Big(\widetilde{D}_0^{(l)}, \widetilde{D}_1^{(l)}\Big)^\dagger$, we understand a column constructed of the absolute terms of Eqs. (7.11) and (7.12). Since $\mathbf{s}_l = \hat{\mathbf{I}}_0 + A_0^{(l)}\mathbf{J}_0\mathbf{t}\mathbf{J}_0^\dagger$ we find

$$
\begin{aligned}
\widetilde{B}_{00}^{(l)} \;&=\; \hat{\mathbf{I}}_0 \,+\, A_0^{(l)}\mathbf{s}_l^{-1}\mathbf{J}_0(M_0 - \mathbf{t})\mathbf{J}_0^\dagger\\
&=\; \mathbf{s}_l^{-1}\Big(\hat{\mathbf{I}}_0 + A_0^{(l)}\mathbf{J}_0\mathbf{t}\mathbf{J}_0^\dagger + A_0^{(l)}\mathbf{J}_0 M_0\mathbf{J}_0^\dagger - A_0^{(l)}\mathbf{J}_0\mathbf{t}\mathbf{J}_0^\dagger\Big)\\
&=\; \mathbf{s}_l^{-1}\Big(\hat{\mathbf{I}}_0 + A_0^{(l)}\mathbf{J}_0 M_0\mathbf{J}_0^\dagger\Big).
\end{aligned}
$$

At the same time

$$\widetilde{B}^{(l)}_{01} = |L_0|\mathbf{s}_l^{-1}\mathbf{J}_0 M_0 \Upsilon\Psi \mathrm{J}_1^\dagger A_1^{(l)}\,,$$
$$\widetilde{B}^{(l)}_{10} = L_1 \mathrm{J}_1 \Psi^* \Upsilon M \mathbf{J}_0^\dagger A_0^{(l)}\,,$$
$$\widetilde{B}^{(l)}_{11} = \hat{I}_1 + L_1 \mathrm{J}_1 \Psi^* U \Psi \mathrm{J}_1^\dagger A_1^{(l)}$$

because $\Upsilon(\Phi + M\Upsilon\Psi) = \Upsilon\mathbf{v}\Psi + \Upsilon M\Upsilon\Psi = (\Upsilon\mathbf{v} + \Upsilon M\Upsilon)\Psi = U\Psi$ (see Sec. 4.).

The absolute terms are

$$\widetilde{D}^{(l)}_0 = |L_0|\,\mathbf{s}_l^{-1}\Big[\mathbf{J}_0(M_0 - \mathbf{t})\Big(\mathbf{I} - A_0^{(l)}\mathbf{J}_0^\dagger\mathbf{s}_l^{-1}\mathbf{J}_0\mathbf{t}\Big) - |L_0|\mathbf{J}_0 M_0\Upsilon\Psi\mathrm{J}_1^\dagger A_1^{(l)}\mathrm{J}_1\Phi^*\Big],$$
$$\widetilde{D}^{(l)}_1 = L_1\mathrm{J}_1\Psi^*\Upsilon M\Big(\mathbf{I} - A_0^{(l)}\mathbf{J}_0^\dagger\mathbf{s}_l^{-1}\mathbf{J}_0\mathbf{t}\Big) - L_1\mathrm{J}_1\Psi^* U\Psi\mathrm{J}_1 A_1^{(l)}\mathrm{J}_1\Phi^*.$$

The operator $\mathbf{s}_l(z)$, $l_0 = \pm 1$ is invertible for all $z \in \mathcal{P}_b$. If $z \notin Z_{\mathrm{res}}$, then $\mathbf{s}_l^{-1}(z)$ is a bounded operator in $\widehat{\mathcal{G}}_0$. Therefore, applying the operator $\mathbf{s}_l$ to both parts of the first equation $\widetilde{B}^{(l)}_{00}\mathbf{X}^{(l)}_0 + \widetilde{B}^{(l)}_{01}\mathbf{X}^{(l)}_1 = \widetilde{D}^{(l)}_0$ of the system $\widetilde{B}^{(l)}\mathbf{X}^{(l)} = \widetilde{D}^{(l)}$, and not changing the second equation, we come to the equivalent system

$$B^{(l)}\mathbf{X}^{(l)} = D^{(l)} \tag{7.12}$$

where

$$B^{(l)} = \begin{pmatrix} \hat{\mathbf{I}}_0 + |L_0|\mathbf{J}_0 M_0\mathbf{J}_0^\dagger A_0^{(l)} & |L_0|\mathbf{J}_0 M_0\Upsilon\Psi\mathrm{J}_1^\dagger A_1^{(l)} \\ L_1\mathrm{J}_1\Psi^*\Upsilon M\mathbf{J}_0^\dagger A_0^{(l)} & \hat{I}_1 + L_1\mathrm{J}_1\Psi^* U\Psi\mathrm{J}_1^\dagger A_1^{(l)} \end{pmatrix}, \tag{7.13}$$

$B^{(l)}(z) : \widehat{\mathcal{G}}_0 \oplus \widehat{\mathcal{H}}_1 \to \widehat{\mathcal{G}}_0 \oplus \widehat{\mathcal{H}}_1$. The absolute term $D^{(l)}$ has components $D^{(l)}_0 = \mathbf{s}_l\widetilde{D}^{(l)}_0$ and $D^{(l)}_1 = \widetilde{D}^{(l)}_1$.

Lemma 7.1. *The inverse operator $\big(B^{(l)}(z)\big)^{-1}$ exists for all $z \in \Pi_l^{(\mathrm{hol})}$ where the inverse operator $S_l^{-1}(z)$ of the truncated three – body scattering matrix $S_l(z)$ given by the first of the equalities in (4.21) exists with $L = \mathrm{diag}\{L_0, L_1\}$, $\tilde{L} = \mathrm{diag}\{|L_0|, L_1\}$, and where the inverse operators $[S_l(z)]_{00}^{-1}$ and $[S_l(z)]_{11}^{-1}$ of $[S_l(z)]_{00} = \hat{I}_0 + \mathrm{J}_0 T\mathrm{J}_0^\dagger A_0 L_0$ and $[S_l(z)]_{11} = \hat{I}_1 + L_1\mathrm{J}_1\Psi^* U\Psi\mathrm{J}_1^\dagger A_1 L_1$, respectively, exist. The components $\big[\big(B^{(l)}(z)\big)^{-1}\big]_{ij}$, $i,j = 0,1$, of the operator $\big(B^{(l)}(z)\big)^{-1}$ admit the representations*

$$\Big[\big(B^{(l)}(z)\big)^{-1}\Big]_{00} = \hat{\mathbf{I}}_0 - \Omega^\dagger\big[S_l^{-1}\big]_{00} \times \big\{|L_0|\mathrm{J}_0 T_0 - [S_l]_{01}[S_l]_{11}^{-1}L_1\mathrm{J}_1\Psi^*\Upsilon M\big\}\,\mathbf{J}_0^\dagger A_0^{(l)}, \tag{7.14}$$

$$\Big[\big(B^{(l)}(z)\big)^{-1}\Big]_{01} = \Omega^\dagger\big[S_l^{-1}\big]_{01}\,, \tag{7.15}$$

$$\Big[\big(B^{(l)}(z)\big)^{-1}\Big]_{10} = -\big[S_l^{-1}\big]_{11}L_1\mathrm{J}_1\Psi^*\Upsilon M\mathbf{J}_0^\dagger A_0^{(l)} \times\Big\{\hat{\mathbf{I}}_0 - \Omega^\dagger[S_l]_{00}^{-1}|L_0|\mathrm{J}_0 T_0\mathbf{J}_0^\dagger A_0^{(l)}\Big\}, \tag{7.16}$$

$$\Big[\big(B^{(l)}(z)\big)^{-1}\Big]_{11} = \big[S_l^{-1}\big]_{00} \tag{7.17}$$

where $T_0 \equiv \Omega M$.

Note that since $|L_0|$ and $A_0^{(l)}$ are numbers which become zero for $l_0 = 0$ simultaneously, the factors $|L_0|$ in (7.14) and (7.16) may be omitted.

Proof. Let us find at the beginning, the components $\Big[\big(B^{(l)}(z)\big)^{-1}\Big]_{00}$ and $\Big[\big(B^{(l)}(z)\big)^{-1}\Big]_{10}$, which will be denoted temporarily (for the sake of brevity) by Y_{00} and Y_{10}. Using Eq. (7.13) we write the system of equations for these components as

$$\big[B^{(l)}\big]_{00} Y_{00} + \big[B^{(l)}\big]_{01} Y_{10} = \hat{\mathbf{I}}_0 \tag{7.18}$$

$$\big[B^{(l)}\big]_{10} Y_{00} + \big[B^{(l)}\big]_{11} Y_{10} = 0\,. \tag{7.19}$$

Eliminating the unknown Y_{10} from the first equation (7.18) with the help of (7.19) we come to the following equation including the element Y_{00} only,

$$\Big\{\hat{\mathbf{I}}_0 + \Omega^\dagger\Big[|L_0|\mathrm{J}_0 T_0 \mathbf{J}_0^\dagger A_0^{(l)} - [S_l]_{01}[S_l]_{11}^{-1} L_1 \mathrm{J}_1 \Psi^* \Upsilon M \mathbf{J}_0^\dagger A_0^{(l)}\Big]\Big\} Y_{00} = \hat{\mathbf{I}}_0\,. \tag{7.20}$$

The operator-matrix on the left−hand side of Eq. (7.20) complementary to $\hat{\mathbf{I}}_0$ has three identical rows. Thus one can apply to Eq. (7.20) the inversion formula

$$\Big[\hat{\mathbf{I}}_0 + \Omega^\dagger(C_1, C_2, C_3)\Big]^{-1} = \hat{\mathbf{I}}_0 - \Omega^\dagger\Big[\hat{I}_0 + C_1 + C_2 + C_3\Big]^{-1}(C_1, C_2, C_3)\,, \tag{7.21}$$

which is true for a wide class of operators C_1, C_2 and C_3. The single essential requirement on C_1, C_2 and C_3 evidently, is the existence of $\big(\hat{I}_0 + C_1 + C_2 + C_3\big)^{-1}$.

In the case concerned

$$C_\beta(z) \equiv \Big\{|L_0|\mathrm{J}_0 T_{0\beta}\mathrm{J}_0^\dagger - [S_l]_{01}[S_l]_{11}^{-1} L_1 \mathrm{J}_1 \Psi^* \Upsilon [M]_\beta \mathrm{J}_0^\dagger\Big\} A_0^{(l)}$$

where $[M]_\beta$ is the β−th column of the matrix M, $[M]_\beta = \big(M_{1\beta}, M_{2\beta}, M_{3\beta}\big)^\dagger$. Thus

$$\begin{aligned}\hat{I}_0 + C_1 + C_2 + C_3 &= \hat{I}_0 + \mathrm{J}_0 T \mathrm{J}_0^\dagger A_0^{(l)} - [S_l]_{01}[S_l]_{11}^{-1}\mathrm{J}_1\Psi^* U_0^\dagger \mathrm{J}_0^\dagger A_0^{(l)} \\ &\equiv [S_l]_{00} - [S_l]_{01}[S_l]_{11}^{-1}[S_l]_{10}.\end{aligned}$$

Note that the components $\big[S_l^{-1}\big]_{ij}$, $i, j = 0, 1$, of S_l^{-1} have the representations

$$\big[S_l^{-1}\big]_{00} = \Big([S_l]_{00} - [S_l]_{01}\,[S_l]_{11}^{-1}\,[S_l]_{10}\Big)^{-1} \tag{7.22}$$

$$\big[S_l^{-1}\big]_{11} = \Big([S_l]_{11} - [S_l]_{10}\,[S_l]_{00}^{-1}\,[S_l]_{01}\Big)^{-1} \tag{7.23}$$

$$\big[S_l^{-1}\big]_{10} = -\,[S_l]_{11}^{-1}\,[S_l]_{10}\,\big[S_l^{-1}\big]_{00} \tag{7.24}$$

$$\big[S_l^{-1}\big]_{01} = -\,[S_l]_{00}^{-1}\,[S_l]_{01}\,\big[S_l^{-1}\big]_{11} \tag{7.25}$$

in terms of the components $[S_l]_{ij}$. It follows from (7.22) that $\hat{I}_0 + C_1 + C_2 + C_3 = \big(\big[S_l^{-1}\big]_{00}\big)^{-1}$. Therefore, in the conditions of the Lemma, the operator $\hat{I}_0 + C_1 + C_2 + C_3$

is invertible. Now, an application of Eq. (7.21) in (7.20) leads us immediately to the representation (7.14) for $\left[\left(B^{(l)}\right)^{-1}\right]_{00}$.

When calculating $Y_{10} = \left[\left(B^{(l)}\right)^{-1}\right]_{10}$ we eliminate from the second equation (7.19) the quantity Y_{00} using Eq. (7.18). In all of this, we need to calculate the inverse operator of $\hat{\mathbf{I}}_0 + \mathbf{J}_0 M_0 \mathbf{J}_0^\dagger A_0^{(l)}$. Here we apply again the relation (7.21) and obtain

$$
\begin{aligned}
\left(\hat{\mathbf{I}}_0 + |L_0|\mathbf{J}_0 M_0 \mathbf{J}_0^\dagger A_0^{(l)}\right)^{-1} &= \left(\hat{\mathbf{I}}_0 + \Omega^\dagger |L_0| \mathrm{J}_0 T_0 \mathbf{J}_0^\dagger A_0^{(l)}\right)^{-1} \\
&= \hat{\mathbf{I}}_0 - \Omega^\dagger \left[S_l\right]_{00}^{-1} |L_0| \mathrm{J}_0 T_0 \mathbf{J}_0^\dagger A_0^{(l)} .
\end{aligned} \tag{7.26}
$$

With the help of (4.21) we can write the resulting equation for Y_{10} as

$$
\begin{aligned}
&\left\{[S_l]_{11} - [S_l]_{10}\,[S_l]_{00}^{-1}\,[S_l]_{01}\right\} Y_{10} \\
&= \; -\,\mathrm{J}_1 \Psi^* \Upsilon M \mathbf{J}_0^\dagger A_0^{(l)} \left[\hat{\mathbf{I}}_0 + \mathbf{J}_0 M_0 \mathbf{J}_0^\dagger A_0^{(l)}\right]^{-1} .
\end{aligned} \tag{7.27}
$$

According to Eq. (7.23) the expression in braces on the left – hand side of Eq. (7.27) coincides with $\left[S_l^{-1}\right]_{11}^{-1}$. Then, from (7.27) we get immediately (7.15).

The system of the equations

$$
\left[B^{(l)}\right]_{00} Y_{01} + \left[B^{(l)}\right]_{01} Y_{11} = 0 \tag{7.28}
$$

$$
\left[B^{(l)}\right]_{10} Y_{01} + \left[B^{(l)}\right]_{11} Y_{11} = \hat{I}_1 \tag{7.29}
$$

for the components $Y_{01} = \left[(B^{(l)})^{-1}\right]_{01}$ and $Y_{11} = \left[(B^{(l)})^{-1}\right]_{11}$ is solved analogously. The search for Y_{11} is a simple problem, since application of the inversion formula (7.27) to Eq. (7.28) immediately gives $Y_{01} = \Omega^\dagger\,[S_l]_{00}^{-1}\,[S_l]_{01}\,Y_{11}$. Substituting this Y_{01} in (7.29) we find

$$
\left\{[S_l]_{11} - [S_l]_{10}\,[S_l]_{00}^{-1}\,[S_l]_{01}\right\} Y_{11} = \hat{I}_1 .
$$

As in Eq. (7.27) the operator on the left – hand side is just $\left[S_l^{-1}\right]_{11}^{-1}$. Inverting it, we come to Eq. (7.17).

When calculating the unknown Y_{01}, we begin by expressing the unknown Y_{11} in terms of it. Using Eq. (7.29) we find

$$
Y_{11} = [S_l]_{11}^{-1}\left(\hat{I}_1 - L_1 \mathrm{J}_1 \Psi \Upsilon M \mathbf{J}_0 A_0^{(l)} Y_{01}\right) . \tag{7.30}
$$

Substituting (7.30) into Eq. (7.28) we obtain an equation with an operator in the position of Y_{01}, which may be inverted with the help of Eq. (7.21). Then we use the chain of equalities

$$
\begin{aligned}
|L_0|\,\mathbf{J}_0 M_0 \Upsilon \Psi \mathrm{J}_1^\dagger A_1^{(l)} &= |L_0|\,\mathbf{J}_0 \Omega^\dagger \Omega M \Upsilon \Psi \mathrm{J}_1^\dagger A_1^{(l)} \\
&= \Omega^\dagger |L_0| \mathrm{J}_0 \Omega M \Upsilon \Psi \mathrm{J}_1^\dagger A_1^{(l)} \\
&= \Omega^\dagger [S_l]_{01} ,
\end{aligned}
$$

simplifying the absolute term as well as the summand on the left−hand side of the equation for Y_{01} appearing there due to (7.30) from the element $\left[B^{(l)}\right]_{01}$. Completing the transformations we find

$$Y_{01} \;=\; -\,\Omega^{\dagger}\Big\{[S_l]_{00} - [S_l]_{01}\,[S_l]_{11}^{-1}\,[S_l]_{10}\Big\}^{-1}[S_l]_{01}[S_l]_{11}^{-1}\,.$$

In view of (7.25), the expression appearing after $\Omega^{\dagger}$ on the right−hand side of the last equation coincides exactly with that for $\left[S_l^{-1}\right]_{01}$. Therefore, we obtain finally Eq. (7.15). Thus, all the components of the inverse operator $\left(B^{(l)}\right)^{-1}$ have already been calculated.

It follows from the representations (7.14) – (7.17) that $\left(B^{(l)}(z)\right)^{-1}$ exists for those $z \in \Pi_l^{(\mathrm{hol})}$ where the inverse operators of $S_l(z)$, $[S_l(z)]_{00}$ and $[S_l(z)]_{11}$ exist. This completes the proof of the Lemma. □

Let us return to Eq. (7.12) and invert the operator $B^{(l)}(z)$ using the relations (7.14) – (7.17). In this way we find the unknowns $\mathbf{X}_0^{(l)}$ and $\mathbf{X}_1^{(l)}$ which express $M^l(z)$ [see Eq. (7.8)].

When carrying out a concrete calculation of $\mathbf{X}_0^{(l)} = \left[\left(B^{(l)}\right)^{-1}\right]_{00} D_0^{(l)} + \left[\left(B^{(l)}\right)^{-1}\right]_{01} D_1^{(l)}$ we use the relation $|L_0|\left[\left(B^{(l)}\right)\right]_{00}\mathbf{J}_0 M_0 = \Omega^{\dagger}|L_0|\left[S_l^{-1}\right]_{00}\mathrm{J}_0 T_0$ that can be checked with the help of (4.21) and (4.8). Along with the identity

$$\mathbf{J}_0\mathbf{t}\Big(\hat{\mathbf{I}}_0 - A_0^{(l)}\mathbf{J}_0^{\dagger}\mathbf{s}_l^{-1}\mathbf{J}_0\mathbf{t}\Big) \;=\; \mathbf{s}_l^{-1}\mathbf{J}_0\mathbf{t}\,, \tag{7.31}$$

this relation simplifies essentially the transform of the product $\left[\left(B^{(l)}\right)^{-1}\right]_{00} D_0^{(l)}$. In addition, when calculating $\mathbf{X}_0^{(l)}$ we use the equalities (7.4). As a result we find

$$\begin{aligned} \mathbf{X}_0^{(l)} \;=\;& \Omega^{\dagger}\big\{|L_0|\left[S_l^{-1}\right]_{00}\mathrm{J}_0 T_0 + \left[S_l^{-1}\right]_{01} L_1\left(\mathrm{J}_1\Psi^*\Upsilon M + \mathrm{J}_1\Phi^*\right)\big\} \\ &-\; |L_0|\,\mathbf{s}_l^{-1}\mathbf{J}_0\mathbf{t}\,. \end{aligned} \tag{7.32}$$

Now, to find $\mathbf{X}_1^{(l)} = \left[\left(B^{(l)}\right)^{-1}\right]_{10} D_0^{(l)} + \left[\left(B^{(l)}\right)^{-1}\right]_{11} D_1^{(l)}$ we observe additionally that the equality $\left\{\hat{\mathbf{I}}_0 - \Omega^{\dagger}\left[S_l^{-1}\right]_{00}^{-1}\mathrm{J}_0 T_0 \mathbf{J}_0^{\dagger} A_0^{(l)}\right\}\mathbf{J}_0 M_0 = \Omega^{\dagger}\left[S_l^{-1}\right]_{00}^{-1}\mathrm{J}_0 T_0$ which simplifies the product $\left[\left(B^{(l)}\right)^{-1}\right]_{10} D_0^{(l)}$ is valid. The final expression for $\mathbf{X}_1^{(l)}$ reads as follows

$$\mathbf{X}_1^{(l)} \;=\; L_1\big\{\left[S_l^{-1}\right]_{10}|L_0|\,\mathrm{J}_0\,T_0 + \left[S_l^{-1}\right]_{11} L_1\mathrm{J}_1\Psi^*\Upsilon M - \left(\hat{I}_1 - \left[S_l^{-1}\right]_{11}\right)L_1\mathrm{J}_1\Psi^*\big\}\,. \tag{7.33}$$

To obtain now a representation for $M^l(z)$, one has only to substitute in Eq. (7.8) the expressions (7.32) for $\mathbf{X}_0^{(l)}$ and (7.33) for $\mathbf{X}_1^{(l)}$. Carrying out a series of simple but rather cumbersome transformations of Eq. (7.8) we arrive as a result at a statement analogous to Theorem 3.2 concerning analytical continuation of the two−body T−matrix. The statement is the following.

Theorem 7.2. *The matrix $M(z)$ admits, in the sense of distributions over $\mathcal{O}(\mathbb{C}^6)$, an analytic continuation in z on the domains $\Pi_l^{(\mathrm{hol})}$ of the unphysical sheets Π_l of the surface $\Re$. The continuation is described by*

$$M^l = M - \left(M\Omega^\dagger \mathrm{J}_0^\dagger, \Phi \mathrm{J}_1^\dagger + M\Upsilon\Psi \mathrm{J}_1^\dagger\right) LA\, S_l^{-1}\tilde{L} \begin{pmatrix} \mathrm{J}_0\Omega M \\ \mathrm{J}_1\Psi^*\Upsilon M + \mathrm{J}_1\Phi^* \end{pmatrix} \tag{7.34}$$

where $S_l(z)$ stands for the truncated scattering matrix (4.21),

$$L = \mathrm{diag}\{l_0, l_{1,1}, \ldots, l_{1,n_1}, l_{2,1}, \ldots, l_{2,n_2}, l_{3,1}, \ldots, l_{3,n_3}\}$$
$$\tilde{L} = \mathrm{diag}\{|l_0|, l_{1,1}, \ldots, l_{1,n_1}, l_{2,1}, \ldots, l_{2,n_2}, l_{3,1}, \ldots, l_{3,n_3}\}.$$

The kernels of all the operators on the right – hand side of Eq. (7.34) *are taken in the physical sheet.*

Note that $LA\, S_l^{-1}(z)\tilde{L} = \tilde{L}\left[S_l^\dagger(z)\right]^{-1} AL$. This means that the relations (7.34) may also be rewritten in terms of the scattering matrices $S_l^\dagger(z)$.

8. Analytic continuation of the scattering matrices

Let $l = \{l_0, l_{1,1}, \ldots, l_{1,n_1}, l_{2,1}, \ldots, l_{2,n_2}, l_{3,1}, \ldots, l_{3,n_3}\}$ with certain l_0, $l_0 = 0$ or $l_0 = \pm 1$, and $l_{\alpha,j}$, $l_{\alpha,j} = 0$ or $l_{\alpha,j} = +1$, $\alpha = 1,2,3,\ j = 1,2,\ldots,n_\alpha$. The truncated scattering matrices $S_l(z) : \widehat{\mathcal{H}}_0 \oplus \widehat{\mathcal{H}}_1 \to \widehat{\mathcal{H}}_0 \oplus \widehat{\mathcal{H}}_1$ and $S_l^\dagger(z) : \widehat{\mathcal{H}}_0 \oplus \widehat{\mathcal{H}}_1 \to \widehat{\mathcal{H}}_0 \oplus \widehat{\mathcal{H}}_1$, given by formulas (4.21), are operator – valued functions of the variable z which are holomorphic in the domain $\Pi_l^{(\mathrm{hol})}$ of the physical sheet Π_0. For $l_0 = 1$ and $l_{\alpha,j} = 1$, $\alpha = 1,2,3,\ j = 1,2,\ldots,n_\alpha$, these matrices coincide with the respective total three – body scattering matrices: $S_l(z) = S(z)$, $S_l^\dagger(z) = S^\dagger(z)$.

We describe now the analytic continuation of the truncated scattering matrices[4)] $S_{l'}(z)$ and $S_{l'}^\dagger(z)$ with a certain multi – index l' in the unphysical sheets $\Pi_l \in \Re$.

We shall use here the representations (7.34) for $M(z)\big|_{\Pi_l}$. As mentioned above, our goal is to find explicit representations for $S_l(z)\big|_{\Pi_{l'}}$ and $S_l^\dagger(z)\big|_{\Pi_{l'}}$ again in terms of the physical sheet.

First, we notice that the function $A_0(z)$ is univalent. It looks like $A_0(z) = -\pi i z^2$ on all the sheets Π_l. At the same time after continuing from Π_0 on Π_l the function $A_{\beta,j}(z) = -\pi i\sqrt{z-\lambda_{\beta,j}}$ keeps its form only if $l_{\beta,j} = 0$. If $l_{\beta,j} = 1$, this function turns into $A'_{\beta,j}(z) = -A_{\beta,j}(z)$. Analogous inversion takes (or does not take) place with arguments $\widehat{P}$, $\widehat{P}'$, $\hat{p}_\alpha$ and $\hat{p}'_\beta$ of the kernels of the operators $\mathrm{J}_0\Omega M\Omega^\dagger \mathrm{J}_0^\dagger$, $\mathrm{J}_0\Omega M\Upsilon\Psi \mathrm{J}_1^\dagger$, $\mathrm{J}_1\Psi^*\Upsilon M\Omega^\dagger \mathrm{J}_0^\dagger$ and $\mathrm{J}_1\Psi^*(\Upsilon \mathbf{v} + \Upsilon M\Upsilon)\Psi \mathrm{J}_1^\dagger$, too. Recall that on the physical sheet Π_0, the action of $\mathrm{J}_0(z)$ $\left(\mathrm{J}_0^\dagger(z)\right)$ transforms $P \in \mathbb{R}^6$ in $\sqrt{z}\widehat{P}$ $\left(P' \in \mathbb{R}^6 \text{ in } \sqrt{z}\widehat{P}'\right)$. At

4) Note that the analytic properties of the truncated scattering matrices or, more exactly, the $(2 \to 2)$ scattering amplitudes in the N – body system with $N \geq 3$ were investigated in the paper [66] in the case of the type (2.3) pair interactions. A proof is given in [66] for existence of analytic continuation of these amplitudes through the cut in vicinities of the branches of the continuous spectrum below the first threshold of the system to breakup into three clusters.

the same time, $p_\alpha \in \mathbb{R}^3$ $\big(p'_\beta \in \mathbb{R}^3\big)$ turns under $\mathrm{J}_{\alpha,i}(z)$ $\big(\mathrm{J}^\dagger_{\beta,j}(z)\big)$ into $\sqrt{z-\lambda_{\alpha,i}}\,\hat{p}_\alpha$ $\big(\sqrt{z-\lambda_{\beta,j}}\,\hat{p}'_\beta\big)$. That is why we introduce the operators $\mathcal{E}(l) = \mathrm{diag}\{\mathcal{E}_0,\, \mathcal{E}_1\}$ where $\mathcal{E}_0$ is the identity operator in $\widehat{\mathcal{H}}_0$ if $l_0 = 0$, and $\mathcal{E}_0$ is the inversion $\big(\mathcal{E}_0 f\big)\big(\widehat{P}\big) = f\big(-\widehat{P}\big)$ if $l_0 = \pm 1$. Analogously, $\mathcal{E}_1(l) = \mathrm{diag}\{\mathcal{E}_{1,1},\, \ldots, \mathcal{E}_{1,n_1}; \mathcal{E}_{2,1},\, \ldots, \mathcal{E}_{2,n_2}; \mathcal{E}_{3,1},\, \ldots, \mathcal{E}_{3,n_3}\}$ where $\mathcal{E}_{\beta,j}$ is the identity operator in $\widehat{\mathcal{H}}^{(\beta,j)}$ if $l_{\beta,j} = 0$ and $\mathcal{E}_{\beta,j}$ is the inversion $(\mathcal{E}_{\beta,j} f)(\hat{p}_\beta) = f(-\hat{p}_\beta)$, if $l_{\beta,j} = 1$. By $\mathrm{e}_1(l)$ we denote the diagonal matrix $\mathrm{e}_1(l) = \mathrm{diag}\{\mathrm{e}_{1,1},\, \ldots, \mathrm{e}_{1,n_1}; \mathrm{e}_{2,1},\, \ldots, \mathrm{e}_{2,n_2}; \mathrm{e}_{3,1},\, \ldots, \mathrm{e}_{3,n_3}\}$ with the elements $\mathrm{e}_{\beta,j} = 1$ if $l_{\beta,j} = 0$ and $\mathrm{e}_{\beta,j} = -1$ if $l_{\beta,j} = 1$. Let $\mathrm{e}(l) = \mathrm{diag}\{\mathrm{e}_0, \mathrm{e}_1\}$ where $\mathrm{e}_0 = +1$.

Theorem 8.1. *If there exists a path on the surface $\Re$ such that while moving along it from the domain $\Pi^{(\mathrm{hol})}_{l'}$ on Π_0 to the domain $\Pi^{(\mathrm{hol})}_{l'} \cap \Pi^{(\mathrm{hol})}_{l'l}$ on Π_l the parameter z stays in intermediate sheets $\Pi_{l''}$ always contained in the domains $\Pi^{(\mathrm{hol})}_{l'} \cap \Pi^{(\mathrm{hol})}_{l''l'}$, then the truncated scattering matrices $S_{l'}(z)$ and $S^\dagger_{l'}(z)$ admit analytic continuation in z on the domain $\Pi^{(\mathrm{hol})}_{l'} \cap \Pi^{(\mathrm{hol})}_{l'l}$ of the sheet Π_l. The continuation is described by*

$$S_{l'}(z)\big|_{\Pi_l} \;=\; \mathcal{E}(l)\Big[\hat{\mathbf{I}} + \tilde{L}'\widehat{\mathcal{T}}L'A\mathrm{e}(l) - \tilde{L}'\widehat{\mathcal{T}}LAS_l^{-1}\tilde{L}\widehat{\mathcal{T}}L'A\mathrm{e}(l)\Big]\mathcal{E}(l)\,, \tag{8.1}$$

$$S^\dagger_{l'}(z)\big|_{\Pi_l} \;=\; \mathcal{E}(l)\Big[\hat{\mathbf{I}} + \mathrm{e}(l)AL'\widehat{\mathcal{T}}\tilde{L}' - \mathrm{e}(l)A\,L\widehat{\mathcal{T}}\tilde{L}\,[S^\dagger_l]^{-1}AL\widehat{\mathcal{T}}\tilde{L}'\Big]\mathcal{E}(l)\,, \tag{8.2}$$

where

$$\begin{aligned} L' &= \big\{l'_0, l'_{1,1},\, \ldots, l'_{1,n_1}, l'_{2,1},\, \ldots, l'_{2,n_2}, l'_{3,1},\, \ldots, l'_{3,n_3}\big\}\,, \\ \tilde{L}' &= \big\{|l'_0|, l'_{1,1},\, \ldots, l'_{1,n_1}, l'_{2,1},\, \ldots, l'_{2,n_2}, l'_{3,1},\, \ldots, l'_{3,n_3}\big\}\,. \end{aligned}$$

Proof. We give the proof for the case of $S_{l'}(z)$. Using the definition (4.8) of the operator $\mathcal{T}(z)$ we rewrite $S_{l'}(z)$ in the form

$$S_{l'}(z) \;=\; \hat{\mathbf{I}} + \tilde{L}'\left[\begin{pmatrix} \mathrm{J}_0\Omega \\ \mathrm{J}_1\Psi^*\Upsilon \end{pmatrix} M\Big(\Omega^\dagger\mathrm{J}^\dagger_0\,,\,\Upsilon\Psi\mathrm{J}^\dagger_1\Big) \;+\; \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{J}_1\Psi^*\Upsilon\mathbf{v}\Psi\mathrm{J}^\dagger_1 \end{pmatrix}\right]L'A\,.$$

Note that when continuing into the sheet $\Pi_{l''}$, the operators $\mathrm{J}_0(z)$, $\mathrm{J}^\dagger_0(z)$, $\mathrm{J}_1(z)$ and $\mathrm{J}^\dagger_1(z)$ turn into $\mathcal{E}_0(l'')\mathrm{J}_0(z)$, $\mathrm{J}^\dagger_0(z)\mathcal{E}_0(l'')$, $\mathcal{E}_1(l'')\mathrm{J}_1(z)$ and $\mathrm{J}^\dagger_1(z)\mathcal{E}_1(l'')$, respectively. At the same time the matrix-function $A(z)$ turns into $A(z)\mathrm{e}(l'')$. Then, using Theorem 7.2 in the domains $\Pi^{(\mathrm{hol})}_{l'} \cap \Pi^{(\mathrm{hol})}_{l'l''}$ of the intermediate sheets $\Pi_{l''}$ we have

$$\begin{aligned} S_{l'}(z)\big|_{\Pi_{l''}} \;=\; & \hat{\mathbf{I}} \,+\, \mathcal{E}(l'')\tilde{L}'\Bigg[\begin{pmatrix} \mathrm{J}_0\Omega \\ \mathrm{J}_1\Psi^*\Upsilon \end{pmatrix} M^{l''}\Big(\Omega^\dagger\mathrm{J}^\dagger_0,\;\; \Upsilon\Psi\mathrm{J}^\dagger_1\Big) \\ & +\, \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{J}_1\Psi^*\Upsilon\mathbf{v}\Psi\mathrm{J}^\dagger_1 \end{pmatrix}\Bigg]L'\mathcal{E}(l'')A\mathrm{e}(l'')\,. \end{aligned} \tag{8.3}$$

Substitution of $M^{l''}(z)$ from (7.34) shows that

$$S_{l'}(z)\big|_{\Pi_{l''}} \;=\; \hat{\mathbf{I}} \,+\, \mathcal{E}(l'')\tilde{L}'\widehat{\mathcal{T}}L'\mathcal{E}(l'')A\mathrm{e}(l'')$$

$$\begin{aligned} &- \mathcal{E}(l'')\tilde{L}'\begin{pmatrix} \mathrm{J}_0\Omega \\ \mathrm{J}_1\Psi^*\Upsilon \end{pmatrix}\Big(M\Omega^\dagger\mathrm{J}_0^\dagger\,,\,[\mathbf{v}+M\Upsilon]\Psi\mathrm{J}_1^\dagger\Big)L''A\,S_{l''}^{-1} \\ &\times\ \tilde{L}''\begin{pmatrix} \mathrm{J}_0\Omega M \\ \mathrm{J}_1\Psi^*[\mathbf{v}+\Upsilon M] \end{pmatrix}\Big(\Omega^\dagger\mathrm{J}_0^\dagger\,,\ \Upsilon\Psi\mathrm{J}_1^\dagger\Big)L'\mathcal{E}(l'')A\mathrm{e}(l'') \end{aligned} \tag{8.4}$$

where the summand immediately following $\hat{\mathbf{I}}$ is generated by the term $M(z)$ of the right – hand side of (7.34). The last summand of (8.4) is comes from the second summand of (7.34).

In view of (7.4) we have $\mathrm{J}_1\Psi^*\mathbf{v}\Omega^\dagger\mathrm{J}_0^\dagger = \mathrm{J}_1\Phi^*\mathrm{J}_0^\dagger\Omega^\dagger = 0$. Analogously, $\mathrm{J}_0\Omega\mathbf{v}\Psi\mathrm{J}_1^\dagger$ is equal to zero, too. Thus, taking into account (4.8) we find

$$\begin{aligned} S_{l'}(z)\big|_{\Pi_{l''}} \ &=\ \hat{\mathbf{I}} + \mathcal{E}(l'')\tilde{L}'\widehat{\mathcal{T}}L'\mathcal{E}(l'')A\mathrm{e}(l'') \\ &\quad -\ \mathcal{E}(l'')\tilde{L}'\widehat{\mathcal{T}}L''A\,S_{l''}^{-1}\tilde{L}''\widehat{\mathcal{T}}L'\mathcal{E}(l'')A\mathrm{e}(l'')\,. \end{aligned} \tag{8.5}$$

By the assumption, the parameter z moves along a path such that in the sheet $\Pi_{l''}$ it is situated in the domain $\Pi_{l'}^{(\mathrm{hol})}\cap\Pi_{l'l''}^{(\mathrm{hol})}$. In this domain, the operators $\big(\tilde{L}'\hat{\mathcal{T}}L'\big)(z)$, $\big(\tilde{L}'\hat{\mathcal{T}}L''\big)(z)$ and $\big(\tilde{L}''\hat{\mathcal{T}}L'\big)(z)$ are defined and depend on z analytically. Consequently, the same may be said also about the function $S_{l'}(z)\big|_{\Pi_{l''}}$. In equal degree, this statement is related to the sheet Π_l. Replacing the values of the multi – index l'' in the representations (8.4) – (8.5) with l, we come to the assertion of the theorem for $S_{l'}(z)\big|_{\Pi_l}$. The validity of the representations (8.2) for $S_{l'}^\dagger(z)\big|_{\Pi_l}$ is established in the same way.□

Remark 8.2. If $l_0 = 0$ then the representation (8.1) for the analytic continuation of $S_l(z)$ into the sheet Π_l (its "own") acquires the simple form [cf. (3.15)]

$$S_l(z)\big|_{\Pi_l}\ =\ \mathcal{E}(l)\Big[\hat{\mathbf{I}}+\mathrm{e}(l)-S_l^{-1}(z)\mathrm{e}(l)\Big]\mathcal{E}(l)\ =\ \mathcal{E}(l)S_l^{-1}(z)\mathcal{E}(l)\,.$$

In the same way $S_l^\dagger(z)\big|_{\Pi_l} = \mathcal{E}(l)\big[S_l^\dagger(z)\big]^{-1}\mathcal{E}(l)$.

9. Representations for the analytic continuation of the resolvent in the unphysical sheets

The resolvent $R(z)$ of the Hamiltonian H for the three – body system is expressed by $M(z)$ according to Eq. (2.11). As we have established, the kernels of all the operators included in the right – hand side of (2.11) admit, in the sense of distributions over $\mathcal{O}(\mathbb{C}^6)$, analytic continuation on the domains $\Pi_l^{(\mathrm{hol})}$ of the unphysical sheets $\Pi_l\subset\Re$. This means that such a continuation is possible as well for the kernel $R(P,P',z)$ of $R(z)$. Moreover, there exists an explicit representation for this continuation analogous to the representation (3.16) for the two – body resolvent.

Theorem 9.1. *The analytic continuation, in the sense of distributions over $\mathcal{O}(\mathbb{C}^6)$, of the resolvent $R(z)$ on the domain $\Pi_l^{(\mathrm{hol})}$ of the unphysical sheet $\Pi_l \subset \Re$ is described by*

$$(9.1)\quad R(z)\big|_{\Pi_l} = R + \big([I - RV]\mathrm{J}_0^\dagger, \Omega[\mathbf{I} - \mathbf{R}_0 M \Upsilon]\Psi \mathrm{J}_1^\dagger\big) LAS_l^{-1}\tilde{L} \begin{pmatrix} \mathrm{J}_0[I - VR] \\ \mathrm{J}_1\Psi^*[\mathbf{I} - \Upsilon M \mathbf{R}_0]\Omega^\dagger \end{pmatrix}.$$

The kernels of all the operators present on the right–hand side of Eq. (9.1) *are taken in the physical sheet.*

Proof. For the analytic continuation $R^l(z)$ of the kernel $R(P, P', z)$ of $R(z)$ into the sheet Π_l we have, according to Eq. (2.11),

$$(9.2)\qquad R^l(z) = R_0^l(z) - R_0^l(z)\Omega M^l(z)\Omega^\dagger R_0^l(z).$$

For $M^l(z)$ we have found already the representation (7.34). Since $R_0^l = R_0 + L_0 A_0 \mathrm{J}_0^\dagger \mathrm{J}_0$ we can rewrite Eq. (9.2) in the form

$$(9.3)\qquad \begin{aligned} R^l &= R_0 - R_0\Omega M^l \Omega^\dagger R_0 + A_0 L_0 \mathrm{J}_0^\dagger\Big(\hat{I}_0 - \mathrm{J}_0\Omega M^l\Omega^\dagger \mathrm{J}_0^\dagger L_0 A_0\Big)\mathrm{J}_0 \\ &\quad - A_0 L_0 \mathrm{J}_0^\dagger \mathrm{J}_0 \Omega M^l \Omega^\dagger R_0 - R_0 \Omega M^l \Omega^\dagger \mathrm{J}_0^\dagger \mathrm{J}_0 L_0 A_0 . \end{aligned}$$

We consider separately the contributions of each summand of (9.4). To this end we shall use the notations

$$\mathbf{B} = \Big(\Omega M \Omega^\dagger \mathrm{J}_0^\dagger, \Omega M \Upsilon \Psi \mathrm{J}_1^\dagger + \Omega\Phi \mathrm{J}_1^\dagger\Big), \quad \mathbf{B}^\dagger = \begin{pmatrix} \mathrm{J}_0 \Omega M \Omega^\dagger \\ \mathrm{J}_1 \Psi^* \Upsilon M \Omega^\dagger + \mathrm{J}_1 \Phi^* \Omega^\dagger \end{pmatrix}.$$

It follows from (7.34) that $\Omega M^l \Omega^\dagger = \Omega M \Omega^\dagger - \mathbf{B}LAS_l^{-1}\tilde{L}\mathbf{B}^\dagger$. Hence, the first two summands of (9.4) give together

$$R_0 - R_0 \Omega M \Omega^\dagger R_0 + R_0 \mathbf{B} LAS_l^{-1}\tilde{L}\mathbf{B}^\dagger R_0 = R + R_0\mathbf{B}LAS_l^{-1}\tilde{L}\mathbf{B}^\dagger R_0 .$$

To transform the third term of (9.4) we again use the representation (7.34). We find

$$\begin{aligned} \mathrm{J}_0\Omega M^l\Omega^\dagger \mathrm{J}_0^\dagger L_0 A_0 &= \widehat{\mathcal{T}}_{00} L_0 A_0 - \Big(\widehat{\mathcal{T}}_{00}, \widehat{\mathcal{T}}_{01}\Big) L\, AS_l^{-1}\,\tilde{L} \begin{pmatrix} \widehat{\mathcal{T}}_{00} \\ \widehat{\mathcal{T}}_{10} \end{pmatrix} L_0 A_0 \\ &= \omega_0 \widehat{\mathcal{T}} LA\omega_0^* - \omega_0 \widehat{\mathcal{T}} LAS_l^{-1}\tilde{L}\widehat{\mathcal{T}} LA\omega_0^* \\ &= \omega_0\widehat{\mathcal{T}} LA\Big(\hat{\mathbf{I}} - S_l^{-1}\tilde{L}\widehat{\mathcal{T}}LA\Big)\tilde{L}\omega_0^* \end{aligned}$$

where ω_0 stands for the projection from $\widehat{\mathcal{H}}_0 \oplus \widehat{\mathcal{H}}_1$ on $\widehat{\mathcal{H}}_0$ defined by $\omega_0\begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = f_0$, $f_0 \in \widehat{\mathcal{H}}_0$, $f_1 \in \widehat{\mathcal{H}}_1$. By ω_0^* we understand, as usual, the adjoint operator of ω_0. Since $S_l = \hat{\mathbf{I}} + \tilde{L}\widehat{\mathcal{T}}LA$ we have $\hat{\mathbf{I}} - S_l^{-1}\tilde{L}\widehat{\mathcal{T}}LA = S_l^{-1}\Big(\hat{\mathbf{I}} + \tilde{L}\widehat{\mathcal{T}}LA - \tilde{L}\widehat{\mathcal{T}}LA\Big) = S_l^{-1}$. Taking in account that $L = L \cdot \tilde{L}$ we find

$$L_0 A_0\big(\hat{I}_0 - \mathrm{J}_0\Omega M^l \Omega^\dagger \mathrm{J}_0^\dagger L_0 A_0\big) = \omega_0 AL\big(\hat{\mathbf{I}} - \tilde{L}\widehat{\mathcal{T}}LAS_l^{-1}\big)\tilde{L}\omega_0^* = \omega_0 LA\, S_l^{-1}\tilde{L}\omega_0^* .$$

This means that the third term of (9.4) may be represented as $\mathrm{J}_0^\dagger\omega_0 LAS_l^{-1}\tilde{L}\omega_0^*$.

When studying the fourth summand of (9.4) we begin by transforming the product $A_0L_0\mathrm{J}_0\Omega M^l\Omega^\dagger$ into a more convenient form. It follows from (7.34) that

$$A_0L_0\mathrm{J}_0\Omega M^l\Omega^\dagger \;=\; A_0L_0\mathrm{J}_0\Omega M\Omega^\dagger \;-\; A_0L_0\Big(\widehat{\mathcal{T}}_{00}\,,\widehat{\mathcal{T}}_{01}\Big)LAS_l^{-1}\tilde{L}\mathbf{B}^\dagger\,.$$

In view of $A_0L_0\mathrm{J}_0\Omega M\Omega^\dagger=\omega_0AL\mathbf{B}^\dagger$ and $A_0L_0\Big(\widehat{\mathcal{T}}_{00},\widehat{\mathcal{T}}_{01}\Big)L=\omega_0AL\widehat{\mathcal{T}}L$ we have

$$A_0L_0\mathrm{J}_0\Omega M^l\Omega^\dagger \;=\; \omega_0\Big(AL-A\,L\widehat{\mathcal{T}}LAS_l^{-1}\tilde{L}\Big)\mathbf{B}^\dagger \;=\; \omega_0LAS_l^{-1}\,\tilde{L}\mathbf{B}^\dagger\,.$$

Analogously, in the fifth term of (9.4) is

$$\Omega M^l\Omega^\dagger\mathrm{J}_0^\dagger L_0A_0 \;=\; \mathbf{B}\tilde{L}\big[S_l^\dagger\big]^{-1}AL\omega_0^* \;=\; \mathbf{B}LAS_l^{-1}\tilde{L}\omega_0^*\,.$$

Thus the last two summands of (9.4) give together

$$-\,\mathrm{J}_0^\dagger\omega_0LAS_l^{-1}\tilde{L}\mathbf{B}^\dagger R_0 \;-\; R_0\mathbf{B}LAS_l^{-1}\tilde{L}\omega_0^*\mathrm{J}_0\,.$$

Substituting these expressions into Eq. (9.4) we find

$$R^l \;=\; R+\big(\mathrm{J}_0^\dagger\omega_0-R_0\mathbf{B}\big)LAS_l^{-1}\tilde{L}\big(\omega_0^*\mathrm{J}_0-\mathbf{B}^\dagger R_0\big)\;.$$

Taking into account the definitions of $\mathbf{B}$ and $\mathbf{B}^\dagger$ as well as the obvious identities $R_0\Omega M\Omega^\dagger=RV$, $R_0\Omega\Phi\mathrm{J}_1=-\Omega\Psi\mathrm{J}_1$ and $\mathrm{J}_1\Phi^*\Omega^\dagger R_0=-\mathrm{J}_1\Psi^*\Omega^\dagger$ we come finally to Eq. (9.1) and this completes the proof. $\square$

10. On the use of the Faddeev differential equations for computations of three – body resonances

It follows from the representations (7.34), (8.1) and (9.1) that the matrices $M(z)\big|_{\Pi_l}$, $S_{l'}(z)\big|_{\Pi_l}$ and the resolvent $R(z)\big|_{\Pi_l}$ may have poles at points belonging to the discrete spectrum $\sigma_d(H)$ of the Hamiltonian H. Nontrivial singularities of $M(z)\big|_{\Pi_l}$, $S_{l'}(z)\big|_{\Pi_l}$ and $R(z)\big|_{\Pi_l}$ correspond to those points $z\in\Pi_0\cap\Pi_l^{(\mathrm{hol})}$ where the inverse truncated scattering matrix $[S_l(z)]^{-1}$ $\big($or $\big[S_l^\dagger(z)\big]^{-1}$ and this is the same$\big)$ does not exist or where it represents an unbounded operator. The points z where $[S_l(z)]^{-1}$ does not exist are the poles of $M(z)\big|_{\Pi_l}$, $S_{l'}(z)\big|_{\Pi_l}$ and $R(z)\big|_{\Pi_l}$. Such points are called (three – body) resonances.

A necessary and sufficient condition [65] for irreversibility of the operator $S_l(z)$ is the existence of a nontrivial solution $\mathcal{A}^{(\mathrm{res})}\in\widehat{\mathcal{H}}_0\oplus\widehat{\mathcal{H}}_1$ of the equation

$$S_l(z)\mathcal{A}^{(\mathrm{res})} \;=\; 0\,. \tag{10.1}$$

The investigation of this equation may be carried out on the basis of the results obtained in Sec. 4 regarding the properties of the kernels of the operator $\widehat{\mathcal{T}}(z)$. We

should postpone this investigation for another paper. Here, we restrict ourselves to the observation that the equation (10.1) may evidently be applied to practical computations of resonances situated in the domains $\Pi_l^{(\mathrm{hol})} \subset \Pi_l$. The resonances have to be considered as those values of $z \in \Pi_0 \cap \Pi_l^{(\mathrm{hol})}$ for which the operators $S_l(z)$ and $S_l^\dagger(z)$ have zero as eigenvalue.

The elements of the scattering matrices $S_l(z)$ and $S_l^\dagger(z)$ are expressed in terms of the amplitudes (continued in z into the physical sheet) for different processes taking place in the three−body system under consideration. The respective formulas [46] rewritten for the components of $\widehat{\mathcal{T}}$, are the following

$$\begin{aligned}
\widehat{\mathcal{T}}_{\alpha,j;\,\beta,k}\big(\hat{p}_\alpha, \hat{p}'_\beta, z\big) &= C_0^{(3)}(z)\mathcal{A}_{\alpha,j;\,\beta,k}\big(\hat{p}_\alpha, \hat{p}'_\beta, z\big)\,,\\
\widehat{\mathcal{T}}_{\alpha,j;\,0}\big(\hat{p}_\alpha, \widehat{P}', z\big) &= C_0^{(3)}(z)\mathcal{A}_{\alpha,j;\,0}\big(\hat{p}_\alpha, \widehat{P}', z\big)\,,\\
\widehat{\mathcal{T}}_{0;\,\beta,\,k}\big(\widehat{P}, \hat{p}'_\beta, z\big) &= C_0^{(6)}(z)\mathcal{A}_{0;\,\beta,k}\big(\widehat{P}, \hat{p}'_\beta, z\big)\,,\\
\widehat{\mathcal{T}}_{00}\big(\widehat{P}, \widehat{P}', z\big) &= C_0^{(6)}(z)\mathcal{A}_{00}\big(\widehat{P}, \widehat{P}', z\big)\,,
\end{aligned}$$

with

$$C_0^{(N)}(z) \;=\; -\,\frac{e^{i\pi(N-3)/4}}{2^{(N-1)/2}\pi^{(N+1)/2}z^{(N-3)/4}}\,,$$

where for the function $z^{(N-3)/4}$ one chooses the main branch. The functions $\mathcal{A}_{\alpha,j;\,\beta,k}$ represent the amplitudes of elastic ($\alpha = \beta$; $j = k$) or inelastic ($\alpha = \beta$; $j \neq k$) scattering and rearrangement ($\alpha \neq \beta$) for the process ($2 \to 2, 3$) in the initial state of which the pair subsystem β is in the k−th bound state and the complementary particle is asymptotically free. The function $\mathcal{A}_{0;\,\beta,k}$ represents in the same process the amplitude of the system for breakup into three separate particles. The amplitudes $\mathcal{A}_{\alpha,j;\,0}$ and $\mathcal{A}_{00}$ correspond, respectively, to the processes ($3 \to 2$) and ($3 \to 3$) beginning from the state with all three particles asymptotically free. Recall that the contributions to $\mathcal{A}_{00}$ from the single and double rescattering represent singular distributions (see Sec. 4).

By describing in Sec. 4 the analytical properties of the matrix $\widehat{\mathcal{T}}$ kernels in the variable z and the smoothness properties in the angular variables $\widehat{P}$ or $\hat{p}_\alpha$ and $\widehat{P}'$ or $\hat{p}'_\beta$, we have in effect described as well the properties of the amplitudes $\mathcal{A}(z)$.

To search for the amplitudes $\mathcal{A}(z)$ continued into the physical sheet, one can use e. g., the formulation [46], [54] of the three−body scattering problem based on the Faddeev differential equations in coordinate space. It is only necessary to come, in this formulation, to complex values of the energy z. The square roots $z^{1/2}$ and $(z - \lambda_{\alpha,j})^{1/2}$, $\alpha = = 1, 2, 3,\ j = 1, 2, \ldots, n_\alpha$, which are present in the formulas of [46], [54] describing asymptotical boundary conditions for the wave function components at infinity, have to be considered as the main branches, i. e., as $\sqrt{z}$ and $\sqrt{z - \lambda_{\alpha,j}}$. Solving the Faddeev differential equations with such conditions one finds in fact the analytical continuation of the wave functions into the physical sheet and, thus, the continuation of the amplitudes $\mathcal{A}(z)$. With the known amplitudes $\mathcal{A}(z)$ one can construct a necessary truncated scattering matrix $S_l(z)$ and find then those values of z for which there exists a nontrivial solution $\mathcal{A}^{(\mathrm{res})}$ of Eq. (10.1). As mentioned above, these values of z represent the three−body resonances in the respective the unphysical sheet Π_l.

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